MAT 5220, Topology Name: ________________ Signature: __________________________
Midterm Exam - In-class Make-up. Due Monday, April 13, 2020.

Your signature attests that you did this exam without reference to text, notes, or other external sources. You may take as long as you like to complete it.

Let X and Y be topological spaces; let A, B, and A_α, $\alpha \in \mathcal{J}$, be subsets of X. Let Int$A$ denote the interior of A, and let $\overline{A}$ denote the closure of A. You may use any of the equivalent definitions of interior and closure that you wish in proving the following:

1. (a) $\text{Int}(A \cap B) = \text{Int}(A) \cap \text{Int}(B)$.

   (b) $\text{Int}(\bigcap A_\alpha) \subseteq \bigcap \text{Int} A_\alpha$.

   (c) Provide a counterexample showing that equality need not hold in the previous statement.

2. (a) $\overline{\bigcap A_\alpha} \subseteq \bigcap \overline{A_\alpha}$

(b) Provide a counterexample showing that equality need not hold in the previous statement. (Hint: an intersection of two sets suffices; equality does not necessarily hold even for finite intersections.)

3. Let τ be a topology on a set X, and let $\mathcal{B} \subseteq \tau$. Prove that $\mathcal{B}$ is a basis for τ if and only if, for every $U \in \tau$ and every $x \in U$, there is a $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

4. Let $f : X \to Y$ be a quotient map. Define an equivalence relation on X by $x \sim x' \Leftrightarrow f(x) = f(x')$. (In your solution, verify that this is an equivalence relation.) Let X^* be the set of equivalence classes, and give X^* the quotient topology induced by X under the natural surjection $x \to [x]$ (where $[x]$ denotes the equivalence class of x). Let $\bar{f} : X^* \to Y$ be defined by $\bar{f}([x]) = f(x)$. (In your solution, explain briefly why $\bar{f}$ is well-defined.) Draw a diagram to illustrate these constructions, and prove that $\bar{f}$ is a homeomorphism.