MAT 4955/5220: Topology **Name:** ______________________________
Final Exam. Due by 5 p.m. on Friday, May 8.

You are expected to work on this exam alone and to refrain from talking about the exam questions to anyone except the professor until the time and date when it is due. You are welcome to help each other learn the relevant material, but not to discuss the specific problems on the exam. You may use your own notes and any published materials that you like. (Cite sources appropriately.)

Your signature constitutes a pledge that you have done the exam according to the above instructions. (Please sign your exam at the top of whatever paper you do it on.)

Signature:______________________________

Solutions must be typeset. Electronic submission, including of this cover page, is fine.

1. Let $\{0,1\}^\infty$ be the subset of $\{0,1\}^\omega$ consisting of all sequences that are eventually zero. Is $\{0,1\}^\infty$ countable or uncountable? Prove your answer.

2. Let $f : X \times Y \to Z$.

   (a) Suppose that for each $x \in X$, the map $f_x : Y \to Z$ defined by $f_x(y) = f(x, y)$ is an open map. Is it necessarily the case that f is an open map? If so, prove it. If not, give a counterexample. (Compare Exercise 12 of Section 18.)

   (b) Suppose that for each $x \in X$, the map $f_x : Y \to Z$ defined by $f_x(y) = f(x, y)$ is a closed map. Is it necessarily the case that f is a closed map? If so, prove it. If not, give a counterexample.

3. Let $p : X \to Y$ and $q : W \to Z$ be quotient maps. Prove or give a counterexample: $p \times q : X \times W \to Y \times Z$ is a quotient map.

4. (a) Prove, for any topological space X, that the connected components of X are closed.

   (b) Prove that if a space X has finitely many connected components, then its connected components are also open.

   (c) Provide an example of a space whose connected components are not open.

5. (a) It is easy to see that a discrete space (that is, a topological space with the discrete topology) is both Hausdorff and locally compact. Prove this!

   (b) Prove that the one-point compactification of $\mathbb{Z}$, the space of integers (in the usual order topology, which is both discrete and the same as the subspace topology as a subspace of the real numbers) is homeomorphic to the subspace $X = \{\frac{1}{k} : k \in \mathbb{Z}\} \cup \{0\}$. (Hint: you just need to show that $\mathbb{Z}$ imbeds in X and that X satisfies the three conditions of Theorem 29.1.)

6. *Extra credit for MAT 4855; required for MAT 5220.* For any group G, the *co-finite topology* is given by the following basis: normal subgroups of finite index are the basic open neighborhoods of the identity, and left translates of these neighborhoods give a neighborhood basis for every other group element. Prove that $\mathbb{Z}$ is Hausdorff in the co-finite topology. (Hint: I suggest approaching this concretely, focusing on $\mathbb{Z}$. Since $\mathbb{Z}$ is commutative, every subgroup is normal. In addition, $\mathbb{Z}$ is infinite cyclic. What do the subgroups of an infinite cyclic subgroup look like? What do their translates look like. (Note: these translates, or cosets, are not in general subrgoups themselves, as you know.)