AXIOMS OF THE REAL NUMBER SYSTEM

Axioms of Addition and Multiplication (Field Axioms)

There exist binary operations $+$ and $\cdot$ on $\mathbb{R}$, such that:

(1) These operations are *associative*:
   (a) $\forall x, y, z \in \mathbb{R}, (x + y) + z = x + (y + z)$.
   (b) $\forall x, y, z \in \mathbb{R}, (x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(2) These operations are *commutative*:
   (a) $\forall x, y \in \mathbb{R}, x + y = y + x$.
   (b) $\forall x, y \in \mathbb{R}, x \cdot y = y \cdot x$.

(3) Each operation has a distinct *identity element*:
   (a) There exists an element $0 \in \mathbb{R}$ such that, $\forall x \in \mathbb{R}$, $x + 0 = x$.
   (b) There exists an element $1 \in \mathbb{R}$ such that $1 \neq 0$ and, $\forall x \in \mathbb{R}$, $x \cdot 1 = x$.

(4) All possible *inverses* exist:
   (a) For each x in $\mathbb{R}$, there exists a y in $\mathbb{R}$ such that $x + y = 0$.
   (b) For each x in $\mathbb{R}$ different from 0, there exists a y in $\mathbb{R}$ such that $x \cdot y = 1$.

(5) The operation $\cdot$ *distributes* over $+$:
   $\forall x, y, z \in \mathbb{R}$, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.

Axioms of the Order with Respect to the Operations

There is a total order relation $<$ on the real number system with the following properties:

(6) The operations respect the order relation:
   (a) If $x > y$, then $x + z > y + z$.
   (b) If $x > y$ and $z > 0$, then $x \cdot z > y \cdot z$.

(7) The order relation $<$ has the *least upper bound property* (to be discussed).