The Logic of Solving an Equation

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$$x = -1 \text{ or } x = 2$$

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$$(x+1)(x-2) = 0$$

$$x = -1 \text{ or } x = 2$$

But wait!

$$-1 \neq \sqrt{-1+2} = \sqrt{1} = 1 \quad \otimes \qquad 2 = \sqrt{2+2} = \sqrt{4} = 2 \quad \surd$$

So $x = 2$ is the only solution.

Clear Exposition and Precise Reasoning is the Goal

How can we explain and justify our solution to the equation?

- How do we describe each step to show what we really did?
- What is the logical relationship among the steps?
- Why did we get the *false* solution $x = -1$?
- How do we know we have *all* the solutions?

A clear, precise, and fully justified solution requires that we use logic!

An Equation is an Open Proposition

- An equation is a *proposition*.
- A *proposition* is a statement that must be true or false.
  - An opinion is *not* a proposition.
    (Example: "Mozart's music is boring." Some would say "true", others would say "false".)
  - A question is *not* a proposition. It is not a statement.
- An equation states that two numbers are equal. Either they are or they are not. The equation must be either true or false (but not both). (Example: $2 = 5$. This is false.)
- An equation involving variables is an *open* proposition.
  - The value of the variable is left open.
  - Whether the statement is true or false depends on the value of the variable.

Propositions Define Sets

- Solving an equation means finding *all* values of the variable that make the equation true.
- The set of all such values is called the *solution set* of the equation. Thus, the solution set of the equation $x = \sqrt{x+2}$ is the set $\{2\}$. More formally:

$$\{x : x = \sqrt{x+2}\} = \{x : x = 2\} = \{2\}.$$

These symbols stand for the (true) statement, "The set of numbers x *such that* $x = \sqrt{x+2}$ is equal to the set of numbers x *such that* $x = 2$, which is (of course) the set containing exactly the element 2." (Two sets are equal if their elements are the same.)

Solving an Equation Requires Logic

- The terms we just discussed may be familiar.
- The explicit reasoning used to solve the equation may not.
- We must first translate our equation of sets,

$$\{x : x = \sqrt{x+2}\} = \{x : x = 2\},$$

into a pair of *conditional* relationships:

1. *If* $x = \sqrt{x+2}$, *then* $x = 2$.
   This means that 2 is the *only possible* solution.
2. *If* $x = 2$, *then* $x = \sqrt{x+2}$.
   This means that 2 really *is* a solution.

Logical Connectives

- *Logical connectives* are words or phrases that convey a relationship between the truth or falsity of propositions.
- You need to know six of them:
  not, and, or, if (commonly used with *then*), *only if, implies*.
- The meanings of "not" and "and" should be clear from common sense. Examples: "$2+2=4$" is true; "$2+2\neq 4$" is false; "$2+2\neq 5$" is true; "$2+2=4$ and $2+3=5$" is true; "$2+2=4$ and $2+2=5$" is false. (The symbol $\neq$ means "does *not* equal".)
- In mathematical language, "or" is not exclusive. It is used as in "blanket or pillow", rather than "coffee or tea": you can have both. Thus, "$2+2=4$ or $2+3=5$" is true; "$2+2=4$ or $2+2=5$" is also true; "$2+2=5$ or $2+2=6$" is false.

Logical Connectives Continued: Conditional Statements

- The connectives "if", "only if", and "implies" are used to relate open propositions.
- The following (true) propositions are synonymous:
  - If $x = 2$, then $x^2 = 4$.
  - $x = 2$ only if $x^2 = 4$.
  - $x = 2$ implies $x^2 = 4$.
- In the above statements, the *condition* $x = 2$ *forces* - *leads to* - the *consequence* $x^2 = 4$.
- *Important note:* Nothing is conveyed about forcing - that is, implication - in the *other* direction. In fact, the statement "If $x^2 = 4$, the $x = 2$ is false. (x could be -2.)

Logical Connectives Continued: Convenient Abbreviations & Symbols

- The following (true) propositions are synonymous:
  - If $x = 2$, then $x^2 = 4$.
  - $x = 2$ only if $x^2 = 4$.
  - $x = 2$ implies $x^2 = 4$.
  - $x = 2 \Rightarrow x^2 = 4$.
- The following (true) propositions are synonymous:
  - If $x = 2$, then $x + 1 = 3$, and if $x + 1 = 3$, then $x = 2$.
  - $x + 1 = 3$ if $x = 2$, and $x + 1 = 3$ only if $x = 2$.
  - $x + 1 = 3$ if and only if $x = 2$.
  - If $x + 1 = 3$, then $x = 2$, and if $x = 2$, then $x + 1 = 3$.
  - $x = 2$ if $x + 1 = 3$, and $x = 2$ only if $x + 1 = 3$.
  - $x = 2$ if and only if $x + 1 = 3$.

Logical Connectives Continued: One More Convenient Symbol

- The following (true) propositions are synonymous:
  - $x = 2 \Rightarrow x + 1 = 3$ and $x + 1 = 3 \Rightarrow x = 2$.
  - $x + 1 = 3 \Rightarrow x = 2$ and $x = 2 \Rightarrow x + 1 = 3$.
  - $x = 2 \Leftrightarrow x + 1 = 3$.
  - $x + 1 = 3 \Leftrightarrow x = 2$.
- The propositions $x + 1 = 3$ and $x = 2$ are *equivalent*.

Practice: Determine Which Propositions are True

1. $2 > 5$ or $2 < 5$.
2. $2 > 5$ and $2 < 5$.
3. $2 < 5$ or $3 < 5$.
4. $2 < 5$ and $3 < 5$.
5. $4 < 5$ and $5 \not< 5$.
6. $x < 5 \Rightarrow x + 1 < 5$.
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Practice: Determine Which Propositions are True

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10	$x + 1 < 6 \Leftrightarrow x < 5$.	True

Equal Sets are Defined by Equivalent Propositions

- We can now summarize our solution to the equation logically and succinctly in symbols:

$$x = \sqrt{x+2} \Leftrightarrow x = 2$$

- The propositions $x = \sqrt{x+2}$ and $x = 2$ are *equivalent*.
- To fully understand *how* we know they are equivalent, we must understand operations in the *real number system*.

What is a Real Number?

- The somewhat misleading name *real* number has nothing to do with the existence of these numbers; all numbers exist as concepts, including the (equally misnamed) imaginary numbers and the complex numbers.
- Historically, the name "real" probably arose because these numbers describe physical quantities such as length, area, volume, and mass. Since complex numbers also play a vital role in describing physical phenomena such as waves, this justification really doesn't hold up.
- The real number system is the smallest system big enough to use for calculus.
- We say *system* because it is not just a set of numbers; there are also operations on the numbers.

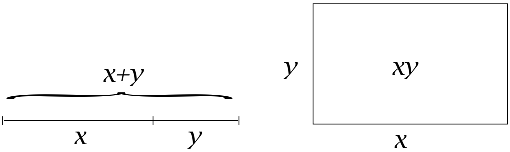
Number operations

- Two basic *operations*: addition and multiplication.
- An *operation* associates an output to a set of inputs in such a way that the inputs *determine* the output. That is, there is *one and only one output* that goes with the inputs.
- As a contrasting example to help explain this, the association of a daughter to a couple is *not* an operation:
  - The couple may not have a daughter; hence, there is no output.
  - The couple may have more than one daughter; hence, the inputs (parents) do not determine a unique output.
- An operation is really just a special name for a basic type of *function*.

Properties of Operations with Real Numbers

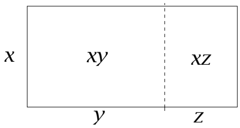
- Addition and multiplication are *binary* operations: they associate an output to *two* inputs.
- Addition and multiplication are *commutative* operations: the order of the inputs does not affect the output. That is:
  - If x and y are (any) real numbers, $x + y = y + x$.
  - If x and y are (any) real numbers, $xy = yx$.
- Addition and multiplication are *associative* operations: the grouping of three or more inputs into pairs does not affect the ultimate output. That is:
  - If x, y, and z are real numbers, $(x + y) + z = x + (y + z)$. For example, $(1 + 2) + 3 = 3 + 3 = 6 = 1 + 5 = 1 + (2 + 3)$
  - If x, y, and z are real numbers, $(xy)z = x(yz)$.
- Since grouping doesn't matter, we can leave it out, simply writing $x + y + z$ and xyz.

Operations Illustrated



Properties of Operations on Real Numbers Continued

- Please note that addition and multiplication are *not* associative in combination. For example, $(1+2)\cdot 3 \neq 1+(2\cdot 3)$.
- However, these operations do have a property in combination: the *distributive* property. Multiplication *distributes* over addition:
  If x, y, and z are real numbers, $x(y+z) = xy + xz$.



$x(y+z) = xy + xz$

Existence of Identities and Inverses: Additive

- The special number 0 is the *additive identity*: Adding 0 to any number gives the same number. That is, if x is any real number, then $x + 0 = x$.
- Every real number has a unique *additive inverse*: When additive inverses are added, the result is 0.
  - Example: $5 + (-5) = 0$.
  - 0 is its own additive inverse: $0 + 0 = 0$.
  - $-(x + y) = (-x) + (-y)$, since $x + y + (-x) + (-y) = x + (-x) + y + (-y) = 0 + 0 = 0$

An Important Consequence

- If x is any real number, then $0x = 0$:
  - $0x = (0+0)x$, since $0+0=0$.
  - $(0+0)x = 0x + 0x$, by the distributive property.
  - Thus, $0x = 0x + 0x$.
- Whatever $0x$ is, it has an additive inverse, $-(0x)$. When we add pairs of equal numbers, we get the same output; hence
  - $0 = 0x + (-(0x)) = 0x + 0x + (-(0x)) = 0x + 0 = 0x$.
  - Thus $0x = 0$

Existence of Identities and Inverses: Multiplicative

- The special number 1 is the *multiplicative identity*: Multiplying any number by 1 gives the same number.
- Every real number *except* 0 has a unique *multiplicative inverse*:
  When multiplicative inverses are multiplied, the result is 1.
  - Example: $(5)\left(\frac{1}{5}\right) = 1$.
  - 1 is its own additive inverse: $(1)(1) = 1$.
  - $\frac{1}{xy} = \left(\frac{1}{x}\right)\left(\frac{1}{y}\right)$, since
    $(xy)\left(\frac{1}{x}\right)\left(\frac{1}{y}\right) = x\left(\frac{1}{x}\right) y \left(\frac{1}{y}\right) = (1)(1) = 1$.
- Zero cannot have multiplicative inverse because, as we have seen, if x is any real number, $0x = 0$. There is *no* real number x for which $0x = 1$; that is impossible.

An Important Consequence

- If $xy = 0$, then $x = 0$ or $y = 0$:
  - We will show that if x is not zero, then y is. This shows that one or the other must be zero.
  - If $x \neq 0$, then it has a multiplicative inverse, $\left(\frac{1}{x}\right)$. Thus:

$$y = 1y = \left(\frac{1}{x}\right) xy = \left(\frac{1}{x}\right) 0 = 0.$$

- We use this fact often when solving quadratic equations!

Defined Operations

- *Subtraction* is defined to be addition of the additive inverse:
  If x and y are real numbers, $x - y = x + (-y)$.
- *Division* is defined to be multiplication by the multiplicative inverse:
  If x and y are real numbers, and $y \neq 0$, $\frac{x}{y} = x\left(\frac{1}{y}\right)$.
- Since 0 has no multiplicative inverse, division by 0 is not defined.
- Many other operations and functions may be defined using the basic ones. For example, x^2 is defined by $x^2 = (x)(x)$. Squaring is a *unitary* operation, since it only takes one input, which is used for both factors.

Using the Properties of Operations

- The properties of addition and multiplication are clear from examples and pictures.
- Why do we specify them in general abstract terms?
- Undoubtedly, you did not need to know the commutative, associative, or distributed property to understand, as a small child, that $2 + 3 = 5$ or that $2 + 3 + 5 = 10$.
- We specify these properties in order to correctly work with expressions involving variables.
- It is easy to calculate with specific numbers, but harder to calculate with unknown numbers.
- Variables represent numbers. The properties we have specified tell us how numbers behave, even if we don't know which specific numbers the variables might represent.

Practice: Simplify Each Expression

1. $$\frac{(2)(3)+3}{3}$$

2. $$\frac{xy+y}{y}$$

Practice: Simplify Each Expression

1. $\dfrac{(2)(3)+3}{3} = \dfrac{(6+3)}{3} = \dfrac{9}{3} = 3$

2. $\dfrac{xy+y}{y}$

Practice: Simplify Each Expression

1. Using instead the multiplicative identity, distributive property, definition of division, and associativity provides a more illuminating method:

$$\frac{(2)(3)+3}{3} = \frac{(2)(3)+(1)(3)}{3} = \frac{(2+1)(3)}{3}$$
$$= \frac{(3)(3)}{3} = (3)(3)\left(\frac{1}{3}\right) = (3)(1) = 3$$

2. $\dfrac{xy+y}{y}$

Practice: Simplify Each Expression

1.
$$\frac{(2)(3)+3}{3} = \frac{(2+1)(3)}{3} = (3)(3)\left(\frac{1}{3}\right) = 3$$

2. While more longwinded than needed for working with specific numbers, this deeper understanding helps us correctly simplify the second expression:
$$\frac{xy+y}{y} = \frac{(x+1)y}{y} = (x+1)(y)\left(\frac{1}{y}\right) = x+1$$

Practice: Simplify Each Expression

3. $\dfrac{2}{3} + \dfrac{3}{5}$

4. $\dfrac{x}{y} + \dfrac{y}{z}$

Practice: Simplify Each Expression

3.
$$\frac{2}{3}+\frac{3}{5}=\left(\frac{2}{3}\right)\left(\frac{5}{5}\right)+\left(\frac{3}{5}\right)\left(\frac{3}{3}\right)=$$
$$(2)(5)\left(\frac{1}{3}\right)\left(\frac{1}{5}\right)+(3)(3)\left(\frac{1}{5}\right)\left(\frac{1}{3}\right)$$
$$=(10+9)\left(\frac{1}{15}\right)=\frac{19}{15}$$

4. $\dfrac{x}{y}+\dfrac{y}{z}$

Practice: Simplify Each Expression

3. $\dfrac{2}{3}+\dfrac{3}{5}=\dfrac{10}{15}+\dfrac{9}{15}=\dfrac{19}{15}$

4. $\dfrac{x}{y}+\dfrac{y}{z}=\dfrac{xz}{yz}+\dfrac{y^2}{yz}=\dfrac{xz+y^2}{yz}$

Practice: Simplify Each Expression

5. $\sqrt{4+9}$

6. $\sqrt{x^2+y^2}$

Practice: Simplify Each Expression

5. $\sqrt{4+9} = \sqrt{13}$

   Note that $\sqrt{4} + \sqrt{9} = 2 + 3 = 5 \neq \sqrt{13}$. $(5^2 = 25.)$

6. $\sqrt{x^2 + y^2}$ cannot be simplified.

Our Example from Algebra: Logical Relationships

- First, let's fill in the logical relationships among the equations.

$$x = \sqrt{x+2} \Rightarrow \tag{1}$$
$$x^2 = x + 2 \Leftrightarrow \tag{2}$$
$$x^2 - x - 2 = 0 \Leftrightarrow \tag{3}$$
$$(x+1)(x-2) = 0 \Leftrightarrow \tag{4}$$
$$x = -1 \text{ or } x = 2 \tag{5}$$

- The relationship between equation (1) and equation (2) is the weak link; since the implication goes only one way, our argument shows only that $x = \sqrt{x+2} \Rightarrow x = -1$ or $x - 2$.

Our Example from Algebra: Justifying our Logic

- Next, we will justify each logical relationship.
- The implication

$$x = \sqrt{x+2} \Rightarrow \tag{1}$$
$$x^2 = x + 2 \tag{2}$$

is true because the expressions on each side of equation (1) represent the same number; therefore, we get the same result on each side when we square that number.

- We cannot assume the reverse implication is true. Recall that $\sqrt{\ }$ means the *positive* square root. If x is negative, $\sqrt{x} = -x$. The converse implication is true for positive values of x, but not for negative values.

Our Example from Algebra: Justifying our Logic

- The biconditional (that is, two-way) implication

$$x^2 = x + 2 \Leftrightarrow \tag{2}$$
$$x^2 - x - 2 = 0 \tag{3}$$

is true because if we add $-x - 2$ to the equal sides of equation (2) we get the same result and, conversely, if we add $x + 2$ to the equal sides of equation (3), we get the same result.

Our Example from Algebra: Justifying our Logic

- The biconditional (that is, two-way) implication

$$x^2 - x - 2 = 0 \Leftrightarrow \tag{3}$$
$$(x+1)(x-2) = 0 \tag{4}$$

  is true because we have simply used the distributive property to rewrite the expression on the left side of equation (3) in a different way.
- We did so in order to use our result, shown earlier, that if the product of two real numbers is zero, then at least one of the numbers must be zero. (The converse, that if one of the two factors is zero, then the product is zero, is also obviously true.)

Our Example from Algebra: Justifying our Logic

- Using the fact that the product of two numbers is zero if and only if one of the two factors is zero, we obtain

$$(x+1)(x-2) = 0 \Leftrightarrow \tag{4}$$

$$x = -1 \text{ or } x = 2 \tag{5}$$

- We now know that -1 and 2 are the only possible solutions, and we have explained why. Since our earlier reasoning showed that the converse of the implication relating equations (1) and (2) is false for negative values of x and true for positive values, we know that -1 is not a solution, but 2 is a solution.
- Thus, we have justified that $x = \sqrt{x+2} \Leftrightarrow x = 2$.