MAT 2550: TWO IMPORTANT THEOREMS ON DIMENSION

1. Preliminaries

Please recall the following fundamental facts, which were proven in class previously. Make sure you understand the ideas behind the proofs, about which I will briefly remind you. We assume throughout that all the vector spaces we are considering are finite dimensional. (There are infinite dimensional vector spaces, but we won't consider them in this course.)

Every finite-dimensional vector space has a well-defined dimension; that is, if there are two finite bases for a vector space, they must contain the same number of vectors. This unique number of called the *dimension* of V, and we will denote it by $\dim V$. The idea of the proof is as follows: replace the vectors in one basis with those in the other, one at a time. Each one you put in is a linear combination of those already there. Because of the linear independence of the vectors you are putting in, you will always be able to take one of the original basis vectors out, and you will not run out of vectors in the first basis before you have replaced them all.

Given a linear transformation $T : V \to W$, there are two important subspaces associated to it. One is the *null space* of T, $\text{Null}T$, defined as $\{v \in V : Tv = \vec{0}\}$. The proof that $\text{Null}T$ is a subspace rests on the linearity of T: a linear combination of vectors that map to the zero vector will map to the same linear combination of zero vectors, hence to zero. The other is the *image* or *range* of T, defined as $\{Tv : v \in V\}$. Our text uses the latter term and the notation $\text{Ran}T$, so I will, too, in order to avoid confusion, although I prefer the former term, and you will see it in the literature as well. $\text{Ran}T$ is simply the space of outputs of T; since any linear combination of outputs is the output of the same linear combination of inputs (again, by the linearity of T), $\text{Ran}T$ is a subspace of W.

Finally, given any collection of vectors $v_1, v_2, v_3, \ldots, v_n \in V$, the *span* of these vectors, denoted $\langle v_1, v_2, v_3, \ldots, v_n \rangle$ is a subspace of V.

Any collection of linearly independent vectors $v_1, v_2, v_3, \ldots, v_n \in V$ can be extended to a basis for V by adding vectors that are not in the span of those already chosen. From any collection of vectors that span V, a basis may be chosen by eliminating vectors that are linear combination of the others.

2. The relation between the dimensions of the null space and range

First, consider as examples the linear maps $\mathbb{R}^2 \to \mathbb{R}^2$ given by the matrices $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. For each of these, sketch the images of the two basis vectors, and also choose a basis

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for the null space and range of each. You should find that the null spaces have dimension $0, 1$, and 2, and the ranges have dimension $2, 1$, and 0, respectively.

Next try the maps $\mathbb{R}^2 \to \mathbb{R}^3$ given by $\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 0 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$. What do you find? I bet you can guess the result of our first major theorem, and perhaps come up with an idea of how to prove it, before you read the proof below.

Theorem. *Given a linear transformation $T : V \to W$, dimNull$T$ + dimRan$T$ = dim$V$.*

Proof. Let $n = \text{dimNull}T$, $k = \text{dim}V$, and $r = k-n$. Choose a basis $v_1, v_2, v_3, \ldots, v_n$ for Null$T$. Extend this to a basis for all of V by adding linearly independent vectors $v_{n+1}, v_{n+2}, v_{n+3}, \ldots, v_{n+r} = v_k$, and let $w_i = v_{n+i}$, for $i = 1, 2, 3, \ldots, r$. We claim that $w_1, w_2, w_3, \ldots, w_r$ is a basis for Ran$T$, from which the desired result clearly follows. We prove the claim in the two usual steps. First, we must show that $\langle w_1, w_2, w_3, \ldots, w_r \rangle = \text{Ran}T$. This is easy! Given $Tv \in W$, we have that $v = \sum_{i=1}^{k} \alpha_i v_i$, since $v_1, v_2, v_3, \ldots, v_k$ is a basis, and in particular a spanning set, for V. Thus, $Tv = T(\sum_{i=1}^{k} \alpha_i v_i) = \sum_{i=1}^{k} \alpha_i T v_i = \sum_{i=n+1}^{k} \alpha_i T v_i = \sum_{i=1}^{r} \alpha_i w_i$, since $Tv_i = \vec{0}$ for $i = 1, 2, 3, \ldots, n$. Next we must show the vectors $w_1, w_2, w_3, \ldots, w_r$ are linearly independent. To that end, suppose $\vec{0} = \sum_{i=1}^{r} \alpha_i w_i = \sum_{i=1}^{r} \alpha_i T v_{n+i} = T(\sum_{i=1}^{r} \alpha_i v_{n+i})$. This means that the vector $\sum_{i=1}^{r} \alpha_i v_{n+i}$ is in Null$T$; therefore, since $v_1, v_2, v_3, \ldots, v_n$ is a basis for Null$T$, there must be coefficients $\beta_1, \beta_2, \beta_3, \ldots, \beta_n$ such that $\sum_{i=1}^{n} \beta_i v_i = \sum_{i=1}^{r} \alpha_i v_{n+i}$. But then $-\beta_1 v_1 - \beta_2 v_2 - \beta_3 v_3 - \cdots - \beta_n v_n + \alpha_1 v_{n+1} + \alpha_2 v_{n+2} + \alpha_3 v_{n+3} + \cdots + \alpha_r v_{n+r} = \vec{0}$. Since the vectors $v_1, v_2, v_3, \ldots, v_k$ are linearly independent, it follows that all the coefficients are zero. In particular, $\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_r = 0$; hence, the vectors $w_1, w_2, w_3, \ldots, w_r$ are linearly independent. □

3. The relationship between the rows and columns of a matrix

The columns of an $m \times n$ matrix A with entries in a scalar field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$, for our purposes) span a subspace of $\mathbb{F}^m$. It is called the *column space* of A, and its dimension is called the *column rank* of A. Similarly, the rows of A span a subspace of $\mathbb{F}^n$ (viewed as row vectors rather than column vectors, of course), called the row space, and its dimension is called the *row rank.* It turns out that the column rank is equal to the row rank for any matrix, and that common dimension is simply called the *rank* of the matrix. There are many different proofs of this fact, some of which use the row operations we will study next; however, Dr. Petrenko and I have just completed a draft of a paper that gives a particularly simple proof. It uses some of the ideas about injective and surjective maps and their one-side inverses that we recently studied.

We know that a function has a inverse applied before it if and only if it is surjective, and an inverse applied after it if and only if it is injective. Thus, a function has a two-sided inverse, which we have shown to be unique, if and only if it is bijective. We showed that the inverse of a bijective linear transformation is automatically linear. That is not the case for inverses that only work on one side. However among the many such inverses, a linear one may be chosen, and that is important for our proof. Here is how to do that.

Suppose a linear transformation $T : V \to W$ is surjective. Given a basis $v_1, v_2, v_3, \ldots, v_k$ for V, the vectors $w_1 = Tv_1$, $w_2 = Tv_2$, $w_3 = Tv_3, \ldots, w_k = Tv_k$ span W. (You should be able to explain why!) Choose a basis from among these vectors, say $w_{i_1} = Tv_{i_1}, w_{i_2} = Tv_{i_2}, w_{i_3} = Tv_{i_3}, \ldots, w_{i_r} = Tvi_r$. Defining $Sw_{i_1} = v_{i_1}, Sw_{i_2} = v_{i_2}, Sw_{i_3} = v_{i_3}, \ldots, Sw_{i_r} = v_{i_r}$ and extending linearly (make sure you know what that means!), we obtain a linear map S such that $TSw = w$ for every $w \in W$. (You should check the details!) Thus we have obtained a linear inverse applied before T.

Suppose a linear transformation $T : V \to W$ is injective. Given a basis $v_1, v_2, v_3, \ldots, v_k$ for V, the vectors $w_1 = Tv_1$, $w_2 = Tv_2$, $w_3 = Tv_3, \ldots, w_k = Tv_k$ are linearly independent. (You should be able to explain why!). Extend this set to a basis $w_1, w_2, w_3, \ldots, w_k, w_{k+1}, \ldots, w_r$ for W. Define

$$Sw_i = \begin{cases} v_i, & \text{for } i = 1, 2, 3, \ldots, k \\ \vec{0}, & \text{for } i = k+1, k+2, \ldots, r \end{cases}.$$

Extend linearly. You can check that $STv = v$ for every $v \in V$, so we have obtained a linear inverse applied after T.

I now refer you to my paper with Dr. Petrenko for the details of the proof that row rank equals column rank.