ROW RANK EQUALS COLUMN RANK: A SIMPLE & ELEMENTARY PROOF

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Abstract. We give a short proof of the classical result that the row rank of a matrix is equal to its column rank. Our proof involves minimal computation and relies on only the most fundamental of notions: dimension, the basic properties of inverse functions, and the symmetry of the identity matrix.

A familiar result in the linear algebra of finite dimensional vector spaces is that the dimension of the vector space spanned by its columns, its *column rank*, is the same as the dimension of the space spanned by its rows, its *row rank*. We present an elementary and accessible reason for this relationship, which relies neither on technical computations nor on an abstract approach to duality. Instead, we introduce a direct approach to dual transformations that makes reference to dual spaces unnecessary for our proof, although the reader may observe that, in what follows, the maps A_{row} and A_{col} are dual to each other.

For other approaches, see for example [1, 3, 4, 5, 6]. In particular, we were inspired by Makiw's short, elegant, and elementary proof [5]; however, it requires a positive-definite inner product. Our proof is valid for matrices over an arbitrary field. Such fields arise, for example, in coding theory, which relies heavily on linear algebra over finite fields.

1. Preliminaries

We recall the following facts, which are likely familiar to many readers.

Let $\mathbb{F}$ be a field. We denote by $\mathbb{F}^l_{\text{col}}$ the space of column vectors with l entries in $\mathbb{F}$, and similarly we denote by $\mathbb{F}^l_{\text{row}}$ the space of row vectors with l entries in $\mathbb{F}$.

Any matrix A with m rows and n columns of entries in $\mathbb{F}$ induces two linear maps:

$$\mathbb{F}^m_{\text{col}} \xleftarrow{A_{\text{col}}} \mathbb{F}^n_{\text{col}}\,,$$

obtained by multiplying the matrix A by column vectors on its right, and

$$\mathbb{F}^m_{\text{row}} \xrightarrow{A_{\text{row}}} \mathbb{F}^n_{\text{row}}\,,$$

obtained by multiplying the matrix A by row vectors on its left. We have purposely denoted these maps with oppositely-directed arrows as a visual device that reflects the side on which subsequent matrices are applied when maps are composed. An obvious, but very useful, observation is that the identity matrix induces the identity map in both directions.

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The map A_{col} is injective if and only if the columns of A are linearly independent, and surjective if and only if they span $\mathbb{F}^m_{\text{col}}$. Similarly, A_{row} is injective if and only if the rows of A are linearly independent, and surjective if and only if they span $\mathbb{F}^n_{\text{row}}$.[1]

Any set of vectors contains a basis for the space it spans. Any set of linearly independent vectors in a space may be extended to a basis for that space. If $v_1, v_2, v_3, \ldots, v_l$ is a basis for a vector space V, and W is a vector space, then any function $\{v_1, v_2, v_3, \ldots, v_l\} \xrightarrow{f} W$ extends uniquely to a linear map $V \xrightarrow{f} W$.

In general, given a function $X \xrightarrow{f} Y$ between any sets X and Y, there exists a function $Y \xrightarrow{g} X$ such that the composition $X \xrightarrow{f} Y \xrightarrow{g} X$ is the identity if and only if f is injective. If $V \overset{f}{\hookrightarrow} W$ is an injective linear map, the desired map $W \xrightarrow{g} V$ may be chosen to be linear by extending (if necessary) the image of a basis for V to a basis for W and defining g arbitrarily on basis vectors not in $f(V)$ (for example, setting $g(w) = 0$ for such vectors). Given a function $Y \xrightarrow{g} X$, there exists a function $X \overset{f}{\hookrightarrow} Y$ such that $X \overset{f}{\hookrightarrow} Y \xrightarrow{g} X$ is the identity if and only if g is surjective.[2] If $W \overset{g}{\twoheadrightarrow} V$ is a surjective linear map, the desired map $V \overset{f}{\hookrightarrow} W$ may be chosen to be linear by selecting a basis for V and defining $f(v)$ to be an arbitrary element of $g^{-1}(v)$ for each basis element v.[3]

2. Proof of the Theorem

Theorem. *The row rank and column rank of any matrix with entries in a field are equal.*

Proof. Let A be a matrix with m rows and n columns, and let k and l denote the column and row ranks of A, respectively. From the columns of A, choose a basis $c_1, c_2, c_3, \ldots, c_k$ for the space its columns span. Let C be the matrix of these columns. Since they are linearly independent, the map C_{col} is injective. Therefore, there is a matrix B such that the composite map $\mathbb{F}^k_{\text{col}} \xtwoheadleftarrow{B_{\text{col}}} \mathbb{F}^m_{\text{col}} \xhookleftarrow{C_{\text{col}}} \mathbb{F}^k_{\text{col}}$ is the identity; hence, $BC = I_k$. But then, applying $BC = I_k$ to row vectors, we have $\mathbb{F}^k_{\text{row}} \xhookrightarrow{B_{\text{row}}} \mathbb{F}^m_{\text{row}} \xtwoheadrightarrow{C_{\text{row}}} \mathbb{F}^k_{\text{row}}$ is the identity; hence C_{row} is surjective. Therefore, at least k rows of C must be linearly independent. Clearly, when the missing coordinates from the rows of A are reinserted, the corresponding rows of A that result remain linearly independent. Thus, $k \leq l$. Applying the argument to the transpose of A yields $l \leq k$, completing the proof. □

References

[1] Cohn, P.M. (1982). *Algebra.*, Vol. 1, 2nd ed., Chichester, England:Wiley.

[2] Cohn, P.M. (1982). *Algebra.* Vol. 2. 2nd ed., Chichester, England:Wiley.

[3] Halmos, P.R. (1958). *Finite-Dimensional Vector Spaces.*, 2nd ed. Princeton, NJ.:D. Van Nostrand.

[4] Hoffman, K., Kunze, R. (1971) *Linear Algebra.* 2nd ed., Englewood Cliffs, NJ:Prentice-Hall.

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[1] These assertions follow immediately for A_{col} from the fact that, if the matrix A has columns $c_1, c_2, c_3, \ldots, c_n$, and v is a column vector with coordinates $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, then $Av = \alpha_1 c_1 + \alpha_2 c_2 + \alpha_3 c_3 + \cdots + \alpha_n c_n$; an obvious modification of the preceding observation establishes the assertions for A_{row}.

[2] For arbitrary sets and functions, this statement is equivalent to the Axiom of Choice [2, Exercise 1 on p. 15]; however, for finite-dimensional vector spaces the Axiom of Choice is unnecessary, since pre-image values need only be chosen for a finite basis.

[3] One can make this choice very efficiently by selecting a basis for V from the image of a basis for W; there will be only finitely many basis vectors in W that map to any vector in this basis.

[5] Mackiw, G. (1995). A note on the equality of the column and row rank of a matrix, *Mathematics Magazine*. 68(4):285-286. http://www.jstor.org/stable/2690576.

[6] Mal'cev, A.I. (1963). *Foundations of Linear Algebra.*. 1st ed. New York, NY:W.H. Freeman.

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