MAT 2550: Review Problems
March 24, 2019

1. Let V and W be vector spaces, and let $L(V, W)$ be the vector space of linear maps from V to W. Scalar multiplication is defined in the obvious way: the linear map $\alpha\lambda : V \to W$ is defined by "$\alpha\lambda$ *of* v is equal to α times λ of v." (I used words because in symbols it looks the same on both sides of the equation! We could use parentheses, but they clutter things up: $(\alpha\lambda)(v) = \alpha(\lambda(v))$.)

   (a) If $\lambda \in L(V, W)$ and $\mu \in L(V, W)$, how is the linear map $\lambda + \mu$ defined?
   The linear map $\lambda + \mu : V \to W$ is defined by $(\lambda + \mu)v =$ _______________________.

   (b) How is the zero vector defined in $L(V, W)$? (It might be easiest to express this in words, too.)

   It should be clear that the operations of addition and scalar multiplication satisfy the axioms of a vector space.

   (c) Let $M(m, n)$ be the space of $n \times m$ matrices. What is the dimension of this space. (Hint: There is an obvious basis, which we have discussed.)

   (d) Suppose V and W are finite-dimensional with dimensions n and m, respectively. What is the dimension of $L(V, W)$? Justify your answer. (Hint: Once bases are chosen for V and W, any linear map from V to W is represented by a matrix; conversely, any matrix represents a linear map. It should be clear – make sure it is – that a linear combination of matrices represents the same linear combination of the maps they represent, so this correspondence is a linear isomorphism.)

   (e) Let Z be a vector space, and let $\kappa : W \to Z$ be a linear map. Verify that composition with κ is a linear map from $L(V, W)$ to $L(V, Z)$. That is, show that the map $\lambda \mapsto \kappa\lambda$ is linear. (Just show that $\kappa(\alpha\lambda + \mu) = \alpha\kappa\lambda + \kappa\mu$ by showing they have the the same output for any vector $v \in V$. Be sure to explain how you use the linearity of κ.)

   (f) Let U be a vector space, and let $\nu : U \to V$ be a linear map. Verify that composition with ν is a linear map from $L(V, W)$ to $L(U, W)$. That is, show that the map $\lambda \mapsto \lambda\nu$ is linear.

2. Extend the set $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ to a basis for $\mathbb{R}^3$. (Note: There are many ways to do this.)

3. Choose a basis for the vector space $\mathbb{P}_2$ of polynomials of degree at most 2 from the set
$$\left\{1 + x, x^2 - x, 2 + x + x^2, 2, 1 + x^2\right\}.$$
   (There are several ways to do this.)

4. Provide a linear left inverse (that is, a post-inverse) map for the map $\lambda : \mathbb{P}_3 \to \mathbb{P}_2$ defined by $\lambda(f) = f'$. Describe this map by indicating its operation on a basis for $\mathbb{P}_2$, and provide its matrix with respect to this basis and the standard bases for $\mathbb{P}_3$. In addition, if you didn't choose the standard basis for $\mathbb{P}_2$, provide its matrix with respect to the standard bases for both $\mathbb{P}_2$ and $\mathbb{P}_3$ as well.

5. Each of the following matrices represents a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ (for some n and m). Determine if this this map is injective, surjective, bijective, or none of these. (Remark: If a basis is chosen for each of two vector spaces V and W with dimensions n and m, respectively, the linear map from V to W the matrix represents will have the same properties with respect to injectivity, surjectivity, and bijectivity. Can you see why? If not, ask!) In addition, for each injective map, provide the matrix of a left inverse, and for each surjective map, provide the matrix of a right inverse. (For bijective maps, these will be the same, and there will be a unique choice; otherwise, there will be more than one correct answer.)

(a) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

6. (a) Prove that the map $\lambda : \mathbb{P}_2 \to \mathbb{P}_2$ defined by $\lambda(f) = f + f'$ is a linear isomorphism. (Hint: just show that the image of the standard basis $\{1, x, x^2\}$ is a basis.) An isomorphism from a space to itself is called an *automorphism*.

   (b) What is the change of basis matrix that converts from the standard basis $\{1, x, x^2\}$ to the basis $\{\lambda(1), \lambda(x), \lambda(x^2)\}$. (Hint: There is an easy way to do this. The matrix for λ in terms of the standard basis converts from the basis $\{\lambda(1), \lambda(x), \lambda(x^2)\}$ to the standard basis, so you just want its inverse. Make sure you can see why! If not, ask!)

   (c) Write the matrix for λ and its inverse (in terms of the standard basis) as products of elementary matrices.

7. Finally, here is an example of how a wise choice of basis helps solve a problem. It is a fact that, through any n points in the plane with distinct x-coordinates, there is a unique polynomial of degree at most $n-1$ whose graph passes through all of them. Here is a method, named after Lagrange but going back to work of Waring and Euler, that finds this polynomial and shows it is unique. For simplicity, we will look at an example in the case $n = 3$. Can you see how to generalize this method to an arbitrary collection of any number of points?

   Consider, for example, the three points $(0, 2)$, $(2, 3)$, and $(4, 1)$.

   (a) Define $l_0(x) = \frac{(x-2)(x-4)}{(0-2)(0-4)} = \frac{(x-2)(x-4)}{8}$, $l_1(x) = \frac{(x-0)(x-4)}{(2-0)(2-4)} = \frac{(x-0)(x-4)}{-4}$, and $l_2 = \frac{(x-0)(x-2)}{8}$. Prove that $\{l_0, l_1, l_2\}$ is a basis for $\mathbb{P}_2$. (Hint: if $\alpha_0 l_0(x) + \alpha_1 l_1(x) + \alpha_2 l_2(x) = 0$, this equation must be true for all values of x. In particular, it must be true for $x = 0, x = 2$, and $x = 4$.)

   (b) Verify that the graph of $2l_0 + 3l_1 + 4l_2$ passes through these three points.