# MAT 2550, Introduction to Linear Algebra: Logical Language, a Primer

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Since you are going to be reading more on your own, I want to make sure that everyone understands the language used in logical reasoning and writing. Logical reasoning involves *propositions*. A proposition is simply a statement with a clear truth value: it is either true or false. The goal of logical reasoning is to establish, based on agreed-upon premises, which propositions are true and which are false. A statement opinion is not a propisition, since it may be true for some people and not for others. In summary, *proposition* is just a fancy word for a statement that *must* be either true or false (even if we don't know which is the case – for example, no one knows whether or not every even integer greater than 2 can be expressed as the sum of two prime numbers, but obviously this proposition, known as Goldbach's Conjecture, must be either true or false).

Goldbach's Conjecture, just mentioned, is an example of a *quantified* proposition: it is a proposition about a set of things, the even integers greater that 2, and it states a property they *all* have, that of being the sum of two primes. If they all have this property, the conjecture is true; if even a single even integer greater than 2 fails to be expressible as the sum of two primes, the conjecture is false. No one has found such a *counter-example* to the conjecture, but we do not have a proof that there is not one out there. The negation of Goldbach's Conjecture, meaning the proposition that is true exactly if the conjecture is false, is that *there exists* an even integer greater than two that is *not* the sum of two primes. In summary, there are two types of quantifiers that can be applied to propositions: *universal*, which states that the proposition holds for all elements of some set, and *existential*, which states that it holds for some element of the set (at least one, possibly more, possibly all, but not necessarily).

Propositions, quantified or not, can be combined to form more complex propositions using logical connectives. There are only a few of these (although a variety of words and phrases are used for each one with the same meaning in colloquial language). We have already seen one: *negation* (indicated by "not," but quantifiers also change, as illustrated above). The others are *conjunction*, indicated by "and" or "but", *disjunction*, indicated by "or," and *implication*, indicated by "if," if ... then," "only if", etc. Disjunctions can be stated as implications, and implications can be state as disjunctions, so logically speaking only one of these connectives is needed, but the connotations they confer (that is, their psychological implications) are different. In other words, a computer can be programmed using either type of connective, but which one is chosen makes a difference to the way people think. Similarly, "and" and "but" mean the same thing logically, but "and" confers the additional connotation of similarity, whereas "but" confers contrast. The following summary should help you understand what I mean. You don't need to know the formal words for these connectives; you need to know how the truth values of the propositions they are composed of affect the truth value of the result. Symbols are often used for all of these connectives, but I will only use the symbols $\Rightarrow$, $\Leftarrow$, and $\Leftrightarrow$, which indicate different implications.

In what follows, think of $p$ and $q$ as arbitrary propositions.

- Conjunction: "$p$ and $q$" is true in the case that both $p$ and $q$ are true. In the case that either or both are false, $p$ and $q$ is false.

- Disjunction: "$p$ or $q$" is true in the case that either or both are true. It is false only in the case that both are false. (So "or" in its mathematical usage is not exclusive – you don't have to choose.)

- Implication: The following propositions mean the same thing: "if $p$, then $q$," "$p$ implies $q$," "$q$ follows from $p$," "$q$ if $p$," "$p$ only if $q$," "$p \Rightarrow q$," "$q \Leftarrow p$." Another helpful way to think of "$p \Rightarrow q$" (and its synonyms) is that $p$ is a *sufficient* condition to make $q$ *necessary* as a consequence. Thus, another name for an implication is a *conditional* statement. "$p \Rightarrow q$" is true in the case that $p$ is false (because $q$ only has to be true if the condition applies) and in the case that $p$ and $q$ are both true; it is false only in the case that $p$ is true but $q$ is false.

  The implication "$p \Rightarrow q$" can also be stated as the disjunction "$q$ or not $p$." Similarly, the disjunction "$p$ or $q$" can be stated as "not $p$ implies q," and also as "not $q$ implies $p$". The extra connotation that a conditional statement adds, which is purely psychological, is to highlight one proposition as the condition from which the other should follow as a consequence. Conditional statements are pervasive in mathematics, because they allow you to assume the condition as a hypothesis in order to focus on proving the consequence as a conclusion under the assumed condition.

  The *biconditional* implication "$p \Leftrightarrow q$" means "$p \Rightarrow q$ and $q \Rightarrow p$." In other words, $p$ and $q$ are equivalent in the sense that each implies the other.

  **Important note:** The negation of "$p \Rightarrow q$" is "$p$ *and not* $q$." Conditional statements are often quantified universally, and a single exception, in which the condition holds but the implies consequence does not, makes the statement false.

  **Important note:** Also note that "$p \Rightarrow q$" means the same thing as its *contrapositive*, "not $q \Rightarrow$ not $p$.