**MAT 2550: Practice End-of-term Exam** **April 27, 2020**

You are expected to work on this exam alone and to refrain from talking about the exam questions to anyone except the professor until the time and date when it is due. You are welcome to help each other learn the relevant material, but not to discuss the specific problems on the exam. You may use your own notes and any published materials that you like. (Cite sources appropriately.)

Your signature constitutes a pledge that you have done the exam according to the above instructions. (Please sign your exam at the top of whatever paper you do it on.)

**Signature:** Solutions

1. Consider the following matrix $A$ and the linear map $\lambda : \mathbb{R}^3 \to \mathbb{R}^3$ it represents when applied to column vectors on its right.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 4 & 1 & 5 \end{pmatrix}$$

(a) Provide a basis for Ran$\lambda$, the range (or image) of $\lambda$.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(b) What is the rank of $A$? (Recall that *rank* = column rank = row rank.) 2

(c) What is the dimension of Null$\lambda$, the null space of this $\lambda$? 1

(d) Provide a basis for Null$\lambda$.

$$A \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} = 0,$$

so $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$ is a basis for Null $\lambda$.

2. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

(a) Compute Det $A$. You should get a non-zero answer, showing that $A$ is invertible.

$$1 \cdot (2 \cdot 0 - 0 \cdot 0) - 0(0 \cdot 0 - 1 \cdot 0) + (-1)(0 \cdot 0 - 1 \cdot 2) = 2$$

(b) Find its inverse matrix, $A^{-1}$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \longleftrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \longleftrightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right], \quad A^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(c) Suppose the matrix $A$ is regarded as a change of basis matrix from basis $\{v_1, v_2, v_3\}$ for $\mathbb{R}^3$ to the standard basis $\{e_1, e_2, e_3\}$. What are $v_1, v_2$, and $v_3$ (in standard coordinates)?

$$V_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}, \quad V_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

(d) What is the change of basis matrix that writes the standard basis $\{e_1, e_2, e_3\}$ in terms of $\{v_1, v_2, v_3\}$?

$A^{-1}$ (see above)

3. (a) Let $\lambda : P_3 \to P_2$ be defined by $\lambda(f) = f'$. Provide the matrix for $\lambda$ in terms of the standard basis $\{1, x, x^2, x^3\}$ for $P_3$ and the standard basis $\{1, x, x^2\}$ for $P_2$.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(b) Is $\lambda$ injective (but not surjective), surjective (but not injective), bijective, or neither? (Circle one, or write one on your paper.)

Injective (only) Surjective (only) Bijective Neither

4. (a) Let $\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by rotation about the origin by 45°. Provide the matrix for $\lambda$ in terms of the standard basis $\{e_1, e_2\}$ for $\mathbb{R}^2$.

$$\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

(b) Is $\lambda$ injective (but not surjective), surjective (but not injective), bijective, or neither? (Circle one, or write one on your paper.)

Injective (only) Surjective (only) Bijective Neither

5. Consider the matrix $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

(a) Find the eigenvalues of this matrix.

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 =$$

$$(\lambda-3)(\lambda-1) = 0 \iff \lambda = 1 \text{ or } \lambda = 3$$

Eigenvalues are 1 and 3.

(b) Find a basis of eigenvectors of this matrix; order it with the eigenvector for the smaller eigenvalue first.

For $\lambda = 1$: $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \longleftrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$: $y$ is free and $x = y$.

So $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ is a basis

For $\lambda = 3$: $\left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right] \longleftrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$, so $x = -y$.

So $\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ is a basis.

(c) If $\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear map represented by the matrix $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ in terms of the standard basis $\{e_1, e_2\}$ for $\mathbb{R}^2$, what is the matrix for $\lambda$ in terms of your (ordered) basis of eigenvectors?

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$