**MAT 2550: Quiz 2** **Name:** ____________________

April 3, 2019

1. Recall that the *trace* (denoted Tr) of a square matrix is the sum of its diagonal entries.

   (a) Using two general $2 \times 2$ matrices $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ with real entries and a general scalar $\alpha \in \mathbb{R}$, show that $\text{Tr}(A + \alpha B) = \text{Tr}A + \alpha \text{Tr}B$.

   (b) Is Tr a linear map from the space of $2 \times 2$ matrices with real entries to $\mathbb{R}$, the space of real scalars? (Please circle correct answer.) **Yes** **No**

   (c) In general, is Tr a linear map from the space of $n \times n$ matrices with real entries to $\mathbb{R}$, the space of real scalars? **Yes** **No**

   (d) Show that Tr: $M_{2\times 2} \to \mathbb{R}$ is surjective, where $M_{2\times 2}$ denotes the space of $2 \times 2$ matrices with real entries. (Hint: Let $\alpha \in \mathbb{R}$ be any real number. What is the trace of $\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$?)

(e) Give a formula for a right (pre-) inverse for Tr: Let $\lambda : \mathbb{R} \to M_{2\times 2}$ be defined by

$$\lambda(\alpha) = \begin{pmatrix} & \\ & \end{pmatrix}.$$

Then $\mathrm{Tr}\lambda(\alpha) = \alpha$ for every $\alpha \in \mathbb{R}$. (Hint: Consider the hint for the previous question! There are many correct answers, though.)

(f) Is Tr: $M_{2\times 2} \to \mathbb{R}$ is injective? Justify your answer.

2. Extend the set $\{1, 1 + x + x^2\}$ to a basis for $\mathbb{P}_2$, the vector space of polynomials of degree at most 2. (There is more than one way to do this.) Verify that your set is a basis. (Hint: Since the dimension of $\mathbb{P}_2$ is 3, it suffices either to show you have three independent vectors or to show you have three vectors that span $\mathbb{P}_2$, whichever is easier. Any independent set of three vectors must span, and any three vectors that span must be independent.)

3. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

(a) Find its inverse matrix, $A^{-1}$.

(b) Write $A^{-1}$ as the product of elementary matrices.

(c) Write $A$ as the product of elementary matrices.

4. Suppose the matrix $A$ is regarded as a change of basis matrix from basis $\{v_1, v_2, v_3\}$ for $\mathbb{R}^3$ to the standard basis $\{e_1, e_2, e_3\}$. What are $v_1, v_2$, and $v_3$ (in standard coordinates)?

5. Let $\lambda : \mathbb{P}_2 \to \mathbb{P}_2$ be defined by $\lambda(f) = f + f'$. Provide the matrix for $\lambda$ in terms of the standard basis $\{1, x, x^2\}$ for $\mathbb{P}_2$.