MAT 2550: End-of-term Exam **Name:** Solutions

April 25, 2018

Be sure to show your reasoning and process for solving problems.

1. Consider the following matrix A and the linear map $\lambda : \mathbb{R}^3 \to \mathbb{R}^3$ it represents when applied to column vectors on its right.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 4 & 1 & 5 \end{pmatrix}$$

(a) Provide a basis for Ranλ, the range (or image) of λ.

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

(b) What is the rank of A? (Recall that *rank* = column rank = row rank.) $\underline{2}$

(c) What is the dimension of Nullλ, the null space of this λ? $\underline{1}$

(d) Provide a basis for Nullλ.

$$\{ e_1 + e_2 - e_3 \} = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\},$$

$$A \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \vec{0}.$$

2. Consider a square $n \times n$ matrix A and a scalar $\alpha \in \mathbb{R}$. Let $D = \text{Det}A$.

(a) Suppose we add α times a column of A to another column of A, obtaining a new matrix A'. What is DetA'? $\underline{D}$

(b) What is the determinant of the elementary matrix E such that $AE = A'$? $\underline{1}$

(c) If $n = 2$, what is Det(αA)? (Recall that αA is the matrix obtained by multiplying every entry of A by α, since its entries are its coordinates. Hint: the answer is *not* αD!) $\underline{\alpha^2 D}$

(d) If $D = \text{Det}A \neq 0$, what is DetA^{-1}? $\underline{1/D}$

3. Consider the following matrix B.

$$B = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

(a) Calculate Det$B$.

$$(1)(-1) - (1)(2) = -3$$

(b) Your answer to (a) should verify that B is invertible. Calculate B^{-1}.

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{array}\right) \xrightarrow[R_2 - 2R_1]{R_2 \to} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -3 & -2 & 1 \end{array}\right) \xrightarrow[R_2/-3]{R_2 \to}$$

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 2/3 & -1/3 \end{array}\right) \xrightarrow[R_1 - R_2]{R_1 \to} \left(\begin{array}{cc|cc} 1 & 0 & 1/3 & 1/3 \\ 0 & 1 & 2/3 & -1/3 \end{array}\right)$$

$$B^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

(c) Write the matrix B^{-1} as a product of elementary matrices.

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1/3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

(d) Write B as a product of elementary matrices.

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

4. Consider the linear map $\lambda : \mathbb{P}_2 \to \mathbb{P}_0 = \mathbb{R}$ given by $\lambda(f) = f''$.

   (a) Write the matrix for λ in terms of the standard basis $\{1, x, x^2\}$ for $\mathbb{P}_2$ and the standard basis $\{1\}$ for $\mathbb{P}_0 = \mathbb{R}$. (Note that it will have only one row.)

   $$\begin{pmatrix} 0 & 0 & 2 \end{pmatrix} \qquad 1'' = 0,\ x'' = 0,\ (x^2)'' = 2$$

   (b) Provide a matrix for a right inverse of λ in terms of these bases. (Note that it will have only one column. There are many correct answers, and it is easy to check if yours is correct!)

   $$\begin{pmatrix} 0 \\ 0 \\ 1/2 \end{pmatrix} \text{ or, generally, } \begin{pmatrix} * \\ ** \\ 1/2 \end{pmatrix}, \text{ where}$$

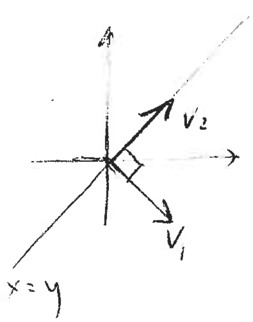
   $*$, $**$ can be anything.

5. Write the matrix that changes from the basis $\{v_1, v_2\} = \left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$ to the standard basis $\{e_1, e_2\} = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ for $\mathbb{R}^2$. (Yes, this is as easy as it looks!)

   $$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Extra Credit! *This problem will also help you check that you understood the previous problem correctly.* Notice that the two vectors $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ of the previous problem are orthogonal (another word for *perpendicular*). This should be easy to see: both make a 45° angle with the positive x-axis.

   (a) Write the matrix for linear map $\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ given by orthogonal projection onto the line $x = y$ in terms of the basis $\{v_1, v_2\}$.



   $$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

   $V_1 \mapsto 0$

   $V_2 \mapsto V_2$, since it is on the line $x=y$.

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(b) Use change of basis matrices to write the matrix for λ in terms of the standard basis for $\mathbb{R}^2$.

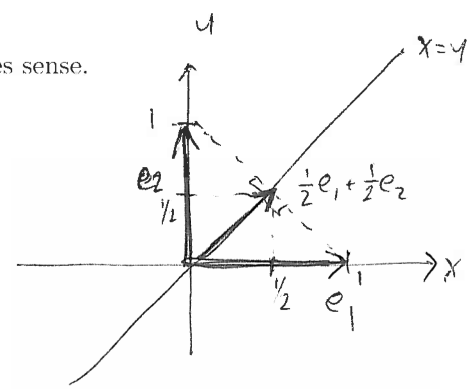
To change $\{e_1, e_2\}$ to $\{v_1, v_2\}$, we need $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}$.

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right) \to$$

$$\left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right),$$ So $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, and we get

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(c) Draw a picture showing why your answer to part (b) makes sense.



(d) Double check by decomposing λ as the composition of a rotation, projection onto the x-axis, and the inverse rotation and seeing that the product of the corresponding matrices agrees with your answer to part (b).

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$ ✓