MAT 2550: End-of-term Exam **Name:** ____________________

April 25, 2018

Be sure to show your reasoning and process for solving problems.

1. Consider the following matrix A and the linear map $\lambda : \mathbb{R}^3 \to \mathbb{R}^3$ it represents when applied to column vectors on its right.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 4 & 1 & 5 \end{pmatrix}$$

   (a) Provide a basis for $\mathrm{Ran}\lambda$, the range (or image) of λ.

   (b) What is the rank of A? (Recall that *rank* = column rank = row rank.) ______

   (c) What is the dimension of $\mathrm{Null}\lambda$, the null space of this λ? ______

   (d) Provide a basis for $\mathrm{Null}\lambda$.

2. Consider a square $n \times n$ matrix A and a scalar $\alpha \in \mathbb{R}$. Let $D = \mathrm{Det}A$.

   (a) Suppose we add α times a column of A to another column of A, obtaining a new matrix A'. What is $\mathrm{Det}A'$? ______

   (b) What is the determinant of the elementary matrix E such that $AE = A'$? ______

   (c) If $n = 2$, what is $\mathrm{Det}(\alpha A)$? (Recall that αA is the matrix obtained by multiplying every entry of A by α, since its entries are its coordinates. Hint: the answer is *not* αD!) ______

   (d) If $D = \mathrm{Det}A \neq 0$, what is $\mathrm{Det}A^{-1}$? ______

3. Consider the following matrix B.

$$B = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

(a) Calculate Det$B$.

(b) Your answer to (a) should verify that B is invertible. Calculate B^{-1}.

(c) Write the matrix B^{-1} as a product of elementary matrices.

(d) Write B as a product of elementary matrices.

4. Consider the linear map $\lambda : \mathbb{P}_2 \to \mathbb{P}_0 = \mathbb{R}$ given by $\lambda(f) = f''$.

   (a) Write the matrix for λ in terms of the standard basis $\{1, x, x^2\}$ for $\mathbb{P}_2$ and the standard basis $\{1\}$ for $\mathbb{P}_0 = \mathbb{R}$. (Note that it will have only one row.)

   (b) Provide a matrix for a right inverse of λ in terms of these bases. (Note that it will have only one column. There are many correct answers, and it is easy to check if yours is correct!)

5. Write the matrix that changes from the basis $\{v_1, v_2\} = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ to the standard basis $\{e_1, e_2\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ for $\mathbb{R}^2$. (Yes, this is as easy as it looks!)

Extra Credit! *This problem will also help you check that you understood the previous problem correctly.* Notice that the two vectors $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ of the previous problem are orthogonal (another word for *perpendicular*). This should be easy to see: both make a 45° angle with the positive x-axis.

   (a) Write the matrix for linear map $\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ given by orthogonal projection onto the line $x = y$ in terms of the basis $\{v_1, v_2\}$.

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(b) Use change of basis matrices to write the matrix for λ in terms of the standard basis for $\mathbb{R}^2$.

(c) Draw a picture showing why your answer to part (b) makes sense.

(d) Double check by decomposing λ as the composition of a rotation, projection onto the x-axis, and the inverse rotation and seeing that the product of the corresponding matrices agrees with your answer to part (b).