

# Vector Algebra & Geometry

Charles Delman

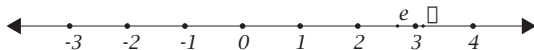
January 12, 2014

## 1 What is a vector?

## 2 Additional Vector Operations

# Real Numbers as Positions

Real numbers may be viewed as positions on a line, once an origin and unit are chosen.



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# Real Numbers as Vectors

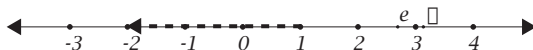
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- In colloquial usage, a *vector* is a vehicle that moves something from one place to another. (For example, the *Anopheles* mosquito is the vector for the malaria parasite.)
- Real numbers may be viewed in this way as well: adding a number *displaces* the position by its *magnitude* in the *direction* of its sign.



$$1 + (-3) = -2$$

# Vectors in Space and in General

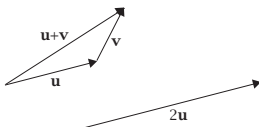
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- The displacement from one point to another in a plane or in a space of higher dimension is a vector. (The distance between the points is the magnitude of that vector.)
- A vector is an object having magnitude and direction.
- Vectors may be added and scaled.



- More abstractly, a *space* is any set whose elements may be added and scaled, and these elements are called vectors.
  - For example, polynomials may be viewed as vectors.
  - However, we will be concerned only with spatial (geometric) vectors.

# Components & Coordinates

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- We typically describe vectors in higher dimensions by breaking them down into simpler components.
- For example, a vector on a map might be viewed as having an East-West component and a North-South component.
- Thus, if  $\mathbf{i}$  is a unit vector (a vector with magnitude one unit) pointing due East and  $\mathbf{j}$  is a unit vector pointing due North, then a displacement of 3 units to the East and 2 units to the South would break down as  $1\mathbf{i} - 2\mathbf{j}$ .
- The vectors  $\mathbf{i}$  and  $\mathbf{j}$  are called a *basis*. Any two vectors that do not lie along the same line may be used as a basis for the plane, and any three vectors that do not all lie in a plane may be used as a basis for three-dimensional space.
- Given a basis, any vector may now be written simply as an ordered sequence of coordinates:  $1\mathbf{i} - 2\mathbf{j} = \langle 1, -2 \rangle$ .

# Standard Rectangular Coordinates

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- We typically assign coordinates in the plane with reference to two perpendicular axes labeled  $x$  and  $y$  and choose the *standard basis* of unit vectors  $\mathbf{i}$  and  $\mathbf{j}$  in the direction of the positive  $x$  and  $y$  axes, respectively.
- We typically assign coordinates in space with reference to three perpendicular axes labeled  $x$ ,  $y$ , and  $z$  and choose the *standard basis* of unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  in the direction of the positive  $x$ ,  $y$ , and  $z$  axes, respectively.
- In terms of the standard basis, the coordinates of a vector are the same as the coordinates of the point obtained by its displacement from the origin.

# Notational Conventions

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You will see both of the following notational conventions:

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle, \mathbf{w} = \langle w_1, w_2, w_3 \rangle$$

(Subscripts go with coordinates, vectors labeled alphabetically.)

$$\mathbf{v}_1 = \langle x_1, y_1, z_1 \rangle, \mathbf{v}_2 = \langle x_2, y_2, z_2 \rangle$$

(Subscripts go with vectors, coordinates labeled alphabetically.)



# Adding & Scaling in Terms of Coordinates

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- It is easy to see that addition of vectors is commutative (draw the picture!) and that scaling distributes over addition (draw the picture!).
- Therefore, adding and scaling is easy in terms of coordinates:

$$\langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$$

$$a\langle x, y, z \rangle = \langle ax, ay, az \rangle$$

(Can you see why this follows? Break the vectors into components.)

# Magnitude in Terms of Standard Coordinates

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- An additional benefit of standard coordinates is that the magnitude of the vector  $\langle x, y, z \rangle$  is given by Pythagorean Theorem:

$$|\langle x, y, z \rangle| = \sqrt{x^2 + y^2 + z^2}.$$

(Draw the picture!)

- *Warning: the basis vectors must be orthogonal unit vectors for this to be the case.*
- A basis of orthogonal unit vectors (such as the standard basis) is called *orthonormal*.

# The *Dot Product*

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- In addition to addition and scalar multiplication, there are additional vector operations of geometric significance.
- We could multiply the magnitudes of vectors, but that is not very meaningful, because the vectors point in different directions. We must also consider the angle between them.
- It is more useful to multiply by the magnitude of the *orthogonal component* of one vector in the direction of the other. Note how this takes the angle between the vectors into account.



# The Dot Product (Continued)

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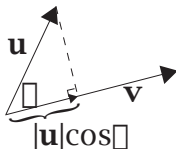
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The geometric procedure just described defines the *dot product*:

$$\mathbf{u} \cdot \mathbf{v} = (|\mathbf{u}| \cos \theta) |\mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \cos \theta$$



The dot product is also often called the *scalar* product, since its output is a number (scalar).

# The Dot Product (Continued)

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- A little trigonometry shows that the dot product is commutative. (Do it!)
- Clearly,  $\mathbf{u} \perp \mathbf{v} \Leftrightarrow \mathbf{u} \cdot \mathbf{v} = 0$ .
- What is not obvious at all is that the dot product is equal to the sum of the products of the coordinates:

$$\langle x_1, y_1, z_1 \rangle \cdot \langle x_2, y_2, z_2 \rangle = x_1x_2 + y_1y_2 + z_1z_2$$

In general, for any dimension  $n$ ,

$$\langle u_1, u_2, u_3, \dots, u_n \rangle \cdot \langle v_1, v_2, v_3, \dots, v_n \rangle = \sum_{i=1}^n u_i v_i.$$

- This formula, which follows from the Law of Cosines, allows us to extract the angle between two vectors easily from their coordinates. (Of course, you have to apply  $\cos^{-1}$  at the end. There is no free lunch! But often just knowing the cosine of the angle is sufficient.)

# Additional Properties of the Dot Product

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- A sum of scalar multiples,  $a\mathbf{u} + b\mathbf{v}$ , is called a *linear combination* of the vectors  $\mathbf{u}$  and  $\mathbf{v}$ ; a vector-valued function is *linear* if the output of a linear combination is that linear combination of the outputs:

$$f(a\mathbf{u} + b\mathbf{v}) = af(\mathbf{u}) + bf(\mathbf{v}).$$

- You should check for yourself that the dot product is commutative ( $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ ) and *bilinear*, meaning that it is *linear* in each input when the other input is held fixed:

$$\begin{aligned}(a\mathbf{u} + b\mathbf{v}) \cdot \mathbf{w} &= a(\mathbf{u} \cdot \mathbf{w}) + b(\mathbf{v} \cdot \mathbf{w}) \\ &= a(\mathbf{w} \cdot \mathbf{u}) + b(\mathbf{w} \cdot \mathbf{v}) = \mathbf{w} \cdot (a\mathbf{u} + b\mathbf{v})\end{aligned}$$

- In standard coordinates,  $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$ .
- In two dimensions,  $\langle -y, x \rangle \perp \langle x, y \rangle$ . (Check!)

# Matrix Notation

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- It is convenient to write the dot product in terms of matrix notation:

$$\mathbf{v} \cdot \mathbf{w} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

(More general matrix multiplication may be defined in terms of the dot product. The entry in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column of the product matrix is the dot product of the  $i^{\text{th}}$  row of the left matrix with the  $j^{\text{th}}$  column of the right matrix. Note that the number of columns of the left matrix must equal the number of rows in the right matrix.)

# Vectors & Flow

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- There is another operation of geometric importance, whose output is a vector.
- To understand its derivation and usefulness, let us begin by considering the idea of *flow*.
- Many physical systems involve flows: water flows down a river or through a pipe, electrons flow through a wire, the energy of the sun flows to the earth.
- Since flow means movement, and vectors describe displacement, it is natural to describe flows using vectors.
- Since a flow is a continual process, we are interested in the *rate* of movement, which can be represented by a vector in the direction of flow whose magnitude represents the speed of flow.



# A Simple Example: Water Flowing at a Constant Rate

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- We first consider a constant rate of flow, so no calculus is involved. Later we will use calculus to analyze flows in which the rate of flow differs from point to point in space or time.
- Suppose water flows at the rate of 2 meters per second down a canal, meaning that each “particle” of water (or, if you like, each point in the water) moves down the canal at this speed.
- We would like to know the rate of *volumetric flow* through a cross-section of the canal: how many cubic meters of water flows through the cross-section each second.

# Flow through a Perpendicular Cross-section

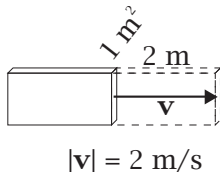
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- The volumetric flow clearly depends on the size and orientation of the cross-section.
- Let us first consider a one-meter-square cross-section perpendicular to the flow.



- In this case, each second a rectangular solid of water of volume 2 cubic meters flows through the section, so the volumetric flow across this surface is  $2\text{ m}^3/\text{s}$ .

# Flow through an Angled Cross-section

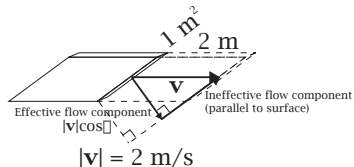
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- More generally consider a one-meter-square cross-section at an angle  $\theta$  to the flow, as shown.



- For  $\theta > 0$  the volumetric flow is less than for a cross-section perpendicular to the flow, because now the mass of water that flows through each second is a parallelepiped with slanted faces and less volume:

$$\frac{dV}{dt} = \frac{2 \cos \theta \text{ m}}{\text{s}} 1 \text{ m}^2 = 2 \cos \theta \text{ m}^3/\text{s}$$

# Flow in Through a Flat Surface (General)

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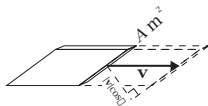
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- In general, the volumetric flow through a flat surface of area  $A$  is

$$\frac{dV}{dt} = (|\mathbf{v}| \cos \theta)A,$$

where  $\mathbf{v}$  is the flow vector (hence  $|\mathbf{v}|$  is the speed of flow) and  $\theta$  is the angle between the surface and the flow vector.



$$|\mathbf{n}| = A$$

$$\mathbf{v} \cdot \mathbf{n} = |\mathbf{n}| |\mathbf{v}| \cos \theta = A |\mathbf{v}| \cos \theta$$

- Let  $\mathbf{n}$  be the vector perpendicular to the surface on the downstream side such that  $|\mathbf{n}| = A$ . Then  $\frac{dV}{dt} = \mathbf{v} \cdot \mathbf{n}$ .

# Geometric Definition of the Cross Product

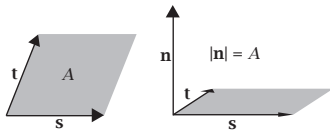
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- Suppose our surface is a parallelogram bounded by two non-collinear vectors  $\mathbf{s}$  and  $\mathbf{t}$ .



- The chosen side of the surface is determined by the order in which  $\mathbf{s}$  and  $\mathbf{t}$  are given in a manner we will describe presently. Let us call it the *positive* side. Thus the vector  $\mathbf{n}$  described above depends only on the ordered pair  $(\mathbf{s}, \mathbf{t})$ ; it is called their *cross product* or *vector product*:

$$\mathbf{s} \times \mathbf{t} = \mathbf{n}$$

# Finding a Formula for the Cross Product: Orientation

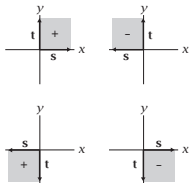
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- We would like a formula for  $\mathbf{n}$  in terms of  $(\mathbf{s}, \mathbf{t})$ .
- Deriving it requires a digression into orientation and determinants.
- First let us describe how a side of the surface is determined by the order of the vectors.
- In keeping with our coordinate conventions, an ordered pair of vectors  $(\mathbf{s}, \mathbf{t})$  determines positive (resp., negative) area if the first is clockwise (resp., counterclockwise) from the second:



(Vectors need not  
be orthogonal.)

# Finding a Formula for the Cross Product: Orientation (Continued)

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- The positive side of the surface is the side you are looking at if the area is positive (and the side facing away from you if the area is negative).
- A simple way to visualize this is “right-hand rule”: a right-handed screw screws forward when you turn it clockwise. The positive side is the direction the screw would go if you turn the first vector toward the second.
- This convention is called “right-hand rule” because, if you curl your right hand (palm in) from the first vector towards the second, your thumb points in the positive direction.)
- An ordered triple of vectors  $(\mathbf{s}, \mathbf{t}, \mathbf{u})$  determines a parallelepiped of positive volume if  $\mathbf{u}$  is on the positive side of the surface determined by  $(\mathbf{s}, \mathbf{t})$ .

# Determinants

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(At this point, make sure you can calculate the determinant of a square matrix using the formula given in the text.)

- The determinant of a  $2 \times 2$  square matrix is the oriented area of the parallelogram bounded by its row vectors, and the determinant of a  $3 \times 3$  square matrix is the oriented volume of the parallelepiped bounded by its row vectors.
- The proof requires a more lengthy development than we have time for, but if you just remember these facts, many properties of the determinant make sense, such as:
  - If two rows of the matrix are the same, the determinant is zero. (The region or solid collapses.)
  - If two rows are interchanged, the determinant changes sign, but has the same magnitude. (The orientation switches but the shape of the region or solid remains the same.)
  - Multilinearity. (Rearrangement of areas or volumes.)



# A Formula for the Cross Product

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- Given vectors  $\mathbf{s} = \langle (s_1, s_2, s_3) \rangle$  and  $\mathbf{t} = \langle (t_1, t_2, t_3) \rangle$ , consider the vector

$$\mathbf{n} = \begin{vmatrix} s_2 & s_3 \\ t_2 & t_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} s_1 & s_3 \\ t_1 & t_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} s_1 & s_2 \\ t_1 & t_2 \end{vmatrix} \mathbf{k}$$

- This calculation may be summarized as the formal determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{vmatrix} = 0$$

(The vectors  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  are not numbers, so it is not a real determinant.)

# A Formula for the Cross Product (Continued)

- The vector  $\mathbf{n} = \begin{vmatrix} s_2 & s_3 \\ t_2 & t_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} s_1 & s_3 \\ t_1 & t_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} s_1 & s_2 \\ t_1 & t_2 \end{vmatrix} \mathbf{k}$  is rigged so that, for any vector  $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ ,

$$\mathbf{u} \cdot \mathbf{n} = \begin{vmatrix} u_1 & u_2 & u_3 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{vmatrix}$$

- In particular,  $\mathbf{n}$  is perpendicular to both  $\mathbf{s}$  and  $\mathbf{t}$ :

$$\mathbf{s} \cdot \mathbf{n} = \begin{vmatrix} s_1 & s_2 & s_3 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{vmatrix} = 0 \qquad \mathbf{t} \cdot \mathbf{n} = \begin{vmatrix} t_1 & t_2 & t_3 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{vmatrix} = 0$$

# A Formula for the Cross Product (Finale!)

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- Furthermore,  $|\mathbf{n}|^2 = \mathbf{n} \cdot \mathbf{n} = \begin{vmatrix} n_1 & n_2 & n_3 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{vmatrix}$ .
- That is,  $|\mathbf{n}|^2$  is the oriented volume of the parallelepiped bounded by  $\mathbf{s}$ ,  $\mathbf{t}$ , and  $\mathbf{n}$ .
- Since  $|\mathbf{n}|^2$  is positive,  $\mathbf{n}$  lies on the positive side of the oriented surface determined by  $(\mathbf{s}, \mathbf{t})$ .
- Furthermore, since  $\mathbf{n}$  is perpendicular to the plane spanned by  $\mathbf{s}$  and  $\mathbf{t}$ , this volume is just  $|\mathbf{n}|A$ , where  $A$  is the (absolute) area of the parallelogram bounded by  $\mathbf{s}$  and  $\mathbf{t}$ .
- Thus  $\mathbf{n}$  satisfies all the geometric properties that define the cross product:

$$\mathbf{s} \times \mathbf{t} = \begin{vmatrix} s_2 & s_3 \\ t_2 & t_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} s_1 & s_3 \\ t_1 & t_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} s_1 & s_2 \\ t_1 & t_2 \end{vmatrix} \mathbf{k}$$