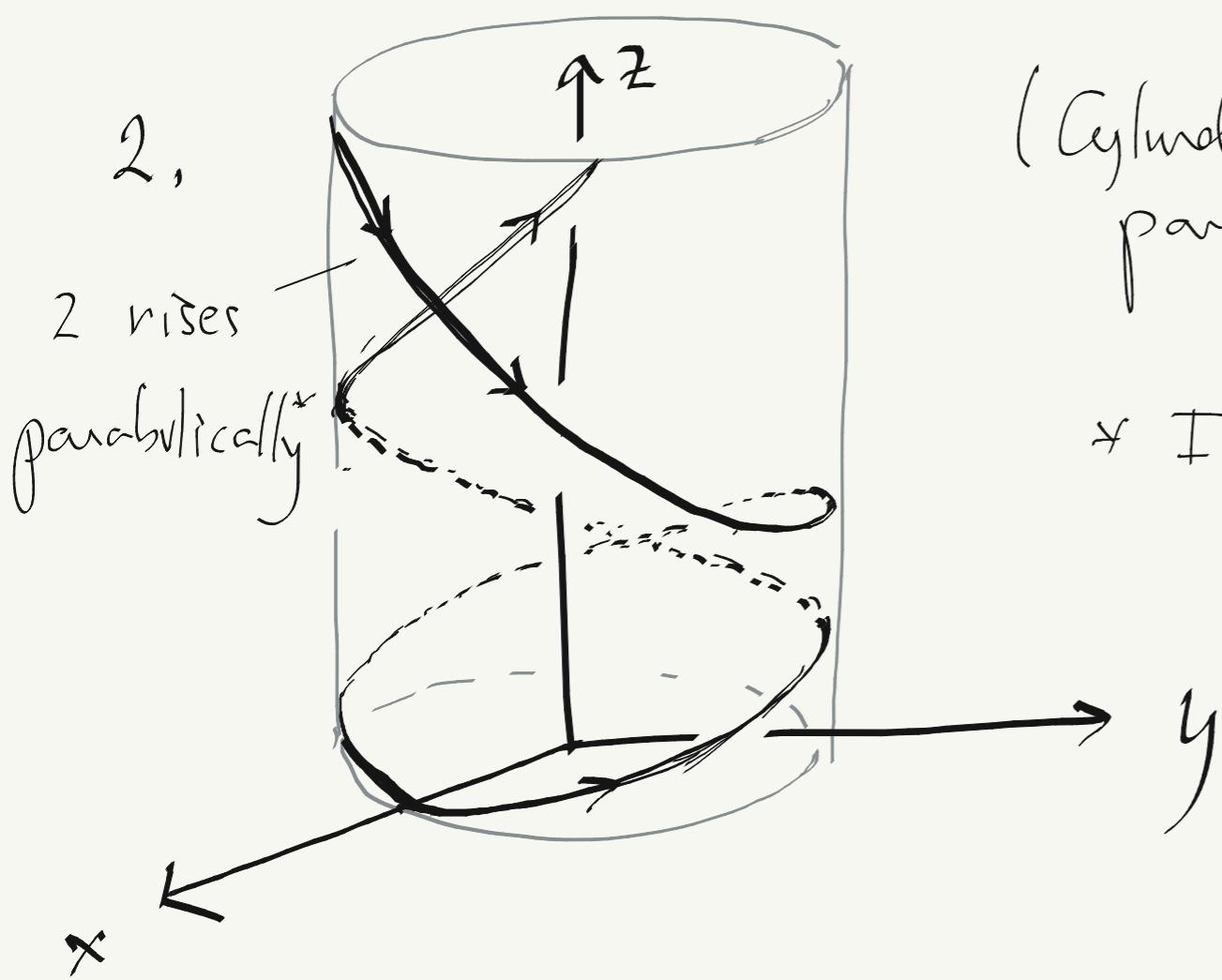


# Solutions to Practice Problems Not Covered in Class. (Note change to problems 2 and 3.)

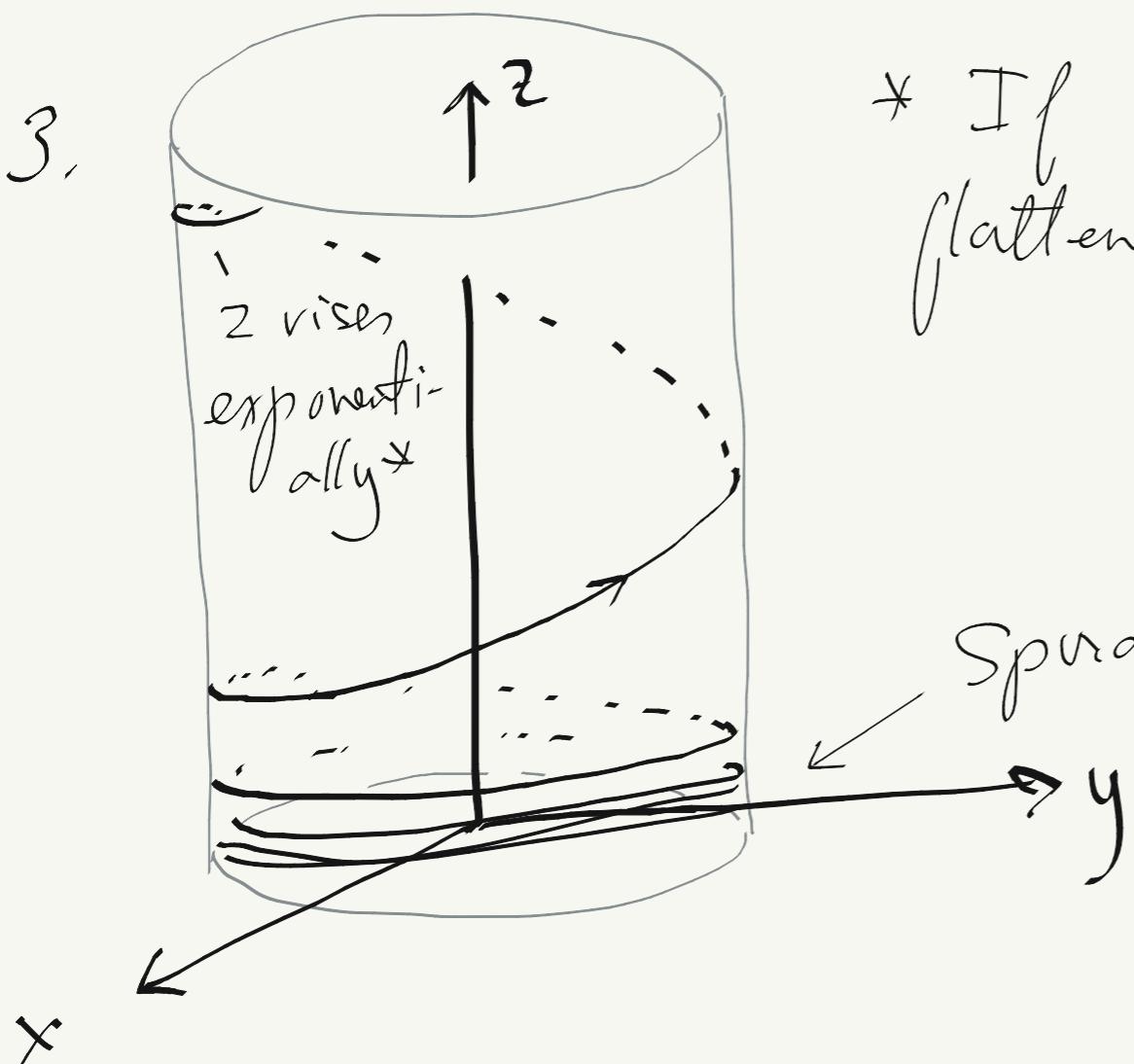
2.



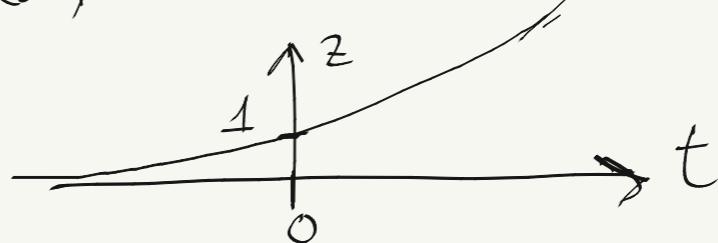
(Cylinder is just a guide, not part of curve.)

- \* If the curve were unwound and flattened, it would be a parabola.

3.



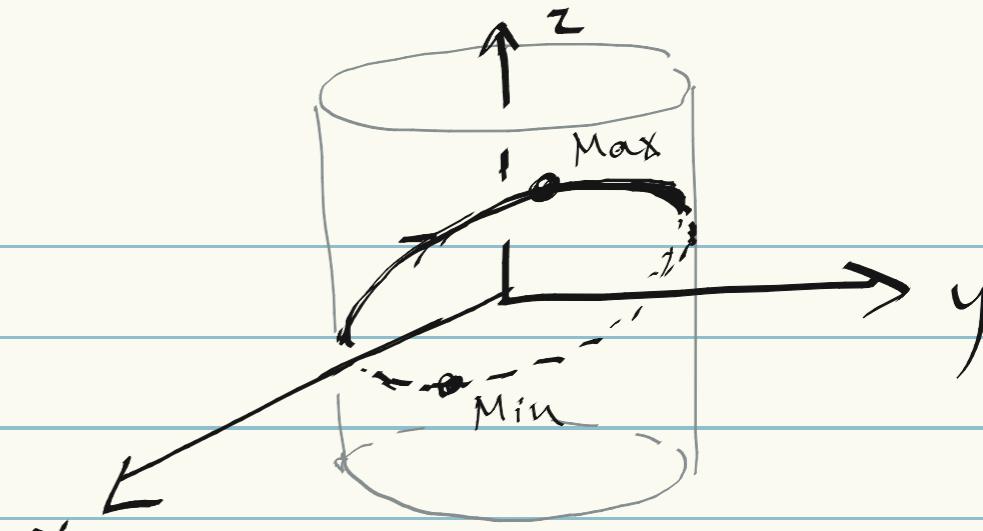
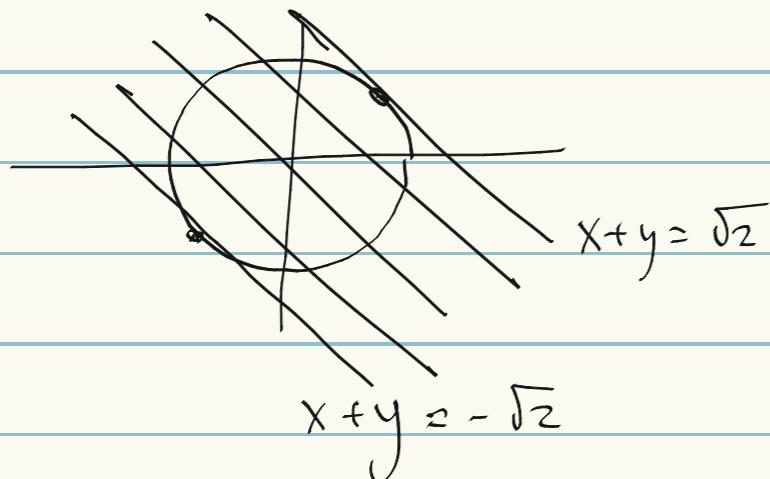
- \* If the curve were unwound and flattened, it would look like this:



Spirals infinitely toward  $z = 0$  as  $t \rightarrow -\infty$ , since  $z = e^t \rightarrow 0$  as  $t \rightarrow -\infty$

4.  $z = x+y$ , so  $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \\ \cos t + \sin t \end{bmatrix}$ .

5. Observe that  $x+y$  is greater or on the curve at  $(\frac{\pi}{2}, \frac{\sqrt{2}}{2})$  and smaller at  $(-\frac{\pi}{2}, -\frac{\sqrt{2}}{2})$ :



(Plane  $z=x+y$  lifts upward along the diagonal, orthogonal to  $\langle 1, 1, -1 \rangle$  and  $\langle -1, -1, 1 \rangle$ .)

7.  $\vec{r}' = \begin{bmatrix} 2t \\ et \\ \cos t \end{bmatrix}$

8. At  $t=0$ ,  $\vec{r}' = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$  and  $|\vec{r}'| = \sqrt{2}$ . So the speed is  $\sqrt{2}$  m/s.

At  $t=\frac{\pi}{2}$ ,  $\vec{r}' = \begin{bmatrix} \pi \\ e^{\pi/2} \\ 0 \end{bmatrix}$ , and  $|\vec{r}'| = \sqrt{\pi^2 + e^{\pi}}$ . So the speed is  $\sqrt{\pi^2 + e^{\pi}}$  m/s.

9. The point  $(0, 1, 0)$  is the position at  $t=0$ . The tangent line is given at unit speed by

$$\vec{r} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{t}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

10. The point  $(\pi^2, e^\pi, 0)$  is the position at  $t=\pi$ . So

$$\vec{r}' = \begin{bmatrix} 2\pi \\ e^\pi \\ -1 \end{bmatrix}, \text{ and the tangent line is parameterized at unit speed by}$$

$$\vec{r} = \begin{bmatrix} \pi^2 \\ e^\pi \\ 0 \end{bmatrix} + \frac{t}{\sqrt{4\pi^2 + e^{2\pi} + 1}} \begin{bmatrix} 2\pi \\ e^\pi \\ -1 \end{bmatrix}$$

11.  $\vec{r}' = \begin{bmatrix} 2t \\ \cos t \end{bmatrix}$ . Sketch was given in class.

12. Referring to  $\vec{r}'$  above, if  $t=0$ ,  $\vec{r}' = \vec{v}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , so the particle is moving at 1 m/s.

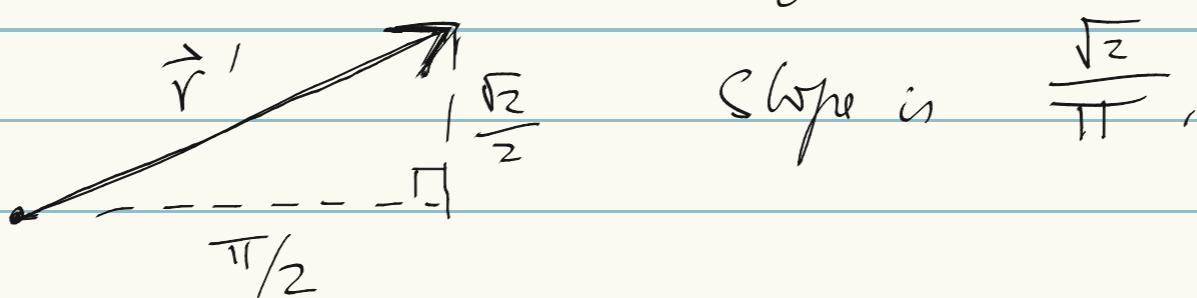
At  $t = \frac{\pi}{2}$ ,  $\vec{r}' = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$ , so the particle is moving at  $\pi$  m/s.

13. At  $t=0$ , the tangent line is clearly  $x=0$ .

At  $t = \frac{\pi}{2}$ , the tangent line is clearly  $y=1$ . This is the value of  $t$  for which the position is  $\begin{bmatrix} \frac{\pi^2}{16} \\ 1 \end{bmatrix}$ .

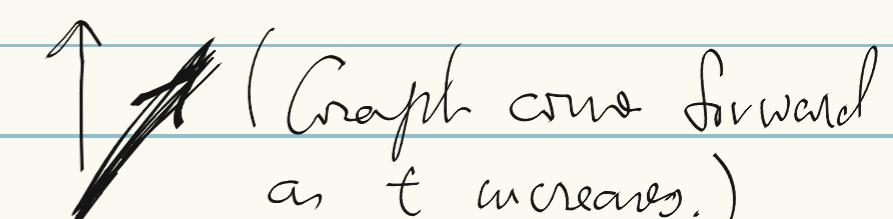
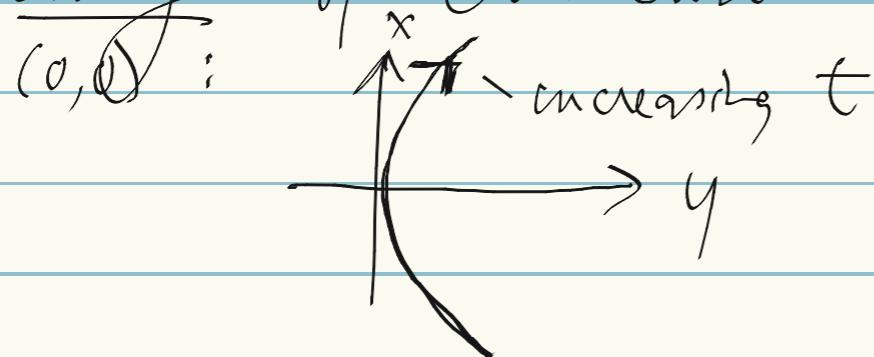
The point  $(\frac{\pi^2}{16}, \frac{\sqrt{2}}{2})$  is the position at  $t = \frac{\pi}{4}$ , so  $\vec{r}' = \begin{bmatrix} \frac{\pi}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$ .

We find the slope of the tangent line as follows:



So the Cartesian equation of the line is  $y - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{\pi} (x - \frac{\pi^2}{16})$ .

14. For the graph, which shows the relationship between inputs and outputs, we need three axes:  $t$ ,  $x$ , and  $y$ . We saw in class that the image of the curve  $\vec{r} = \begin{bmatrix} t^2 \\ \sin t \end{bmatrix}$  looks like this near  $(0,0)$ :

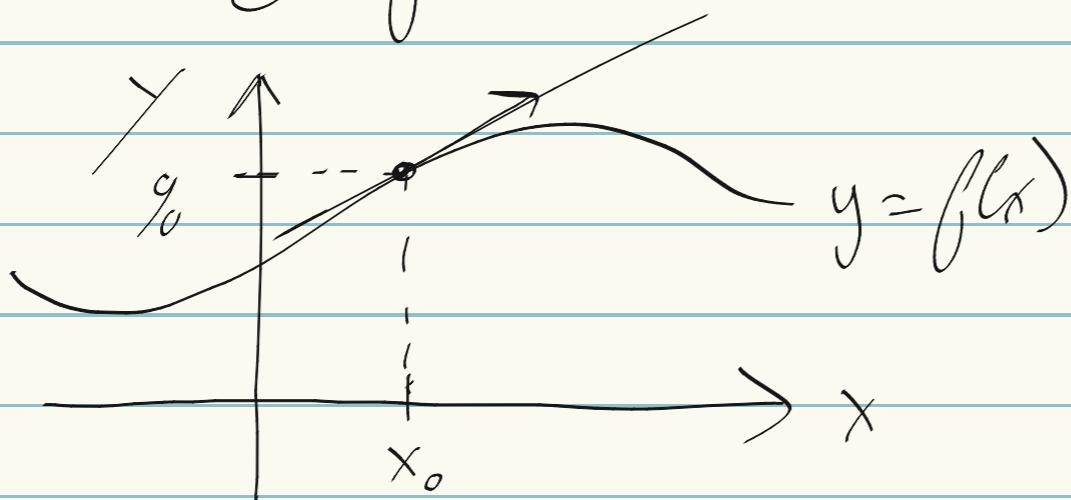


So the graph looks like this:

$$\text{The parametric equation of the graph is } \vec{r} = \begin{bmatrix} t \\ x \\ y \end{bmatrix} = \begin{bmatrix} t \\ t^2 \\ \sin t \end{bmatrix}, \text{ so}$$

its derivative is  $\vec{r}' = \begin{bmatrix} 2t \\ \cos t \end{bmatrix}$ .

To understand this better, let's re-examine the case  $y = f(x)$  (one output :  $y$ )



The Cartesian equation for the tangent line to the graph of

$$y = f(x)$$
 is  $y - y_0 = f'(x_0)(x - x_0)$ .

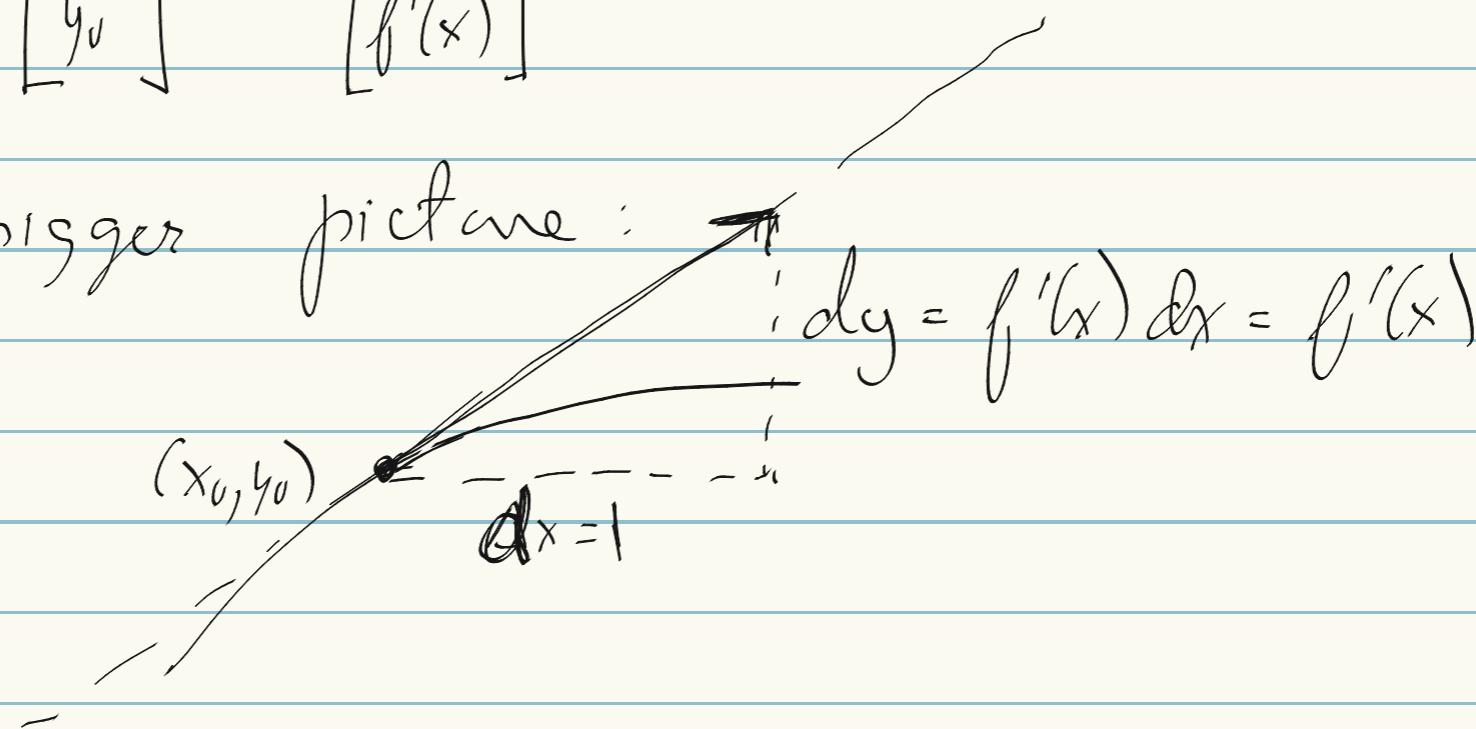
Parametrically, we use  $x$  as the parameter, and the curve is

$$\vec{r} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ f(x) \end{bmatrix}. \text{ So } \vec{r}' = \begin{bmatrix} 1 \\ f'(x) \end{bmatrix}, \text{ and}$$

a parametric equation of the tangent line at  $(x_0, y_0)$  is

$$\vec{r} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t \begin{bmatrix} 1 \\ f'(x_0) \end{bmatrix}.$$

Here's a bigger picture:



With two outputs that depend on  $t$ ,  $t$  is the independent variable. The graph of

$$\vec{r} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$$

is the parametric curve

$$\vec{r} = \begin{bmatrix} t \\ x \\ y \end{bmatrix} = \begin{bmatrix} t \\ f(t) \\ g(t) \end{bmatrix}, \text{ so } \vec{r}' = \begin{bmatrix} 1 \\ f'(t) \\ g'(t) \end{bmatrix}.$$

Now, in our example,  $x = t^2$  and  $y = \sin t$ , so

The graph has derivative  $\vec{r}' = \begin{bmatrix} 1 \\ 2t \\ \cos t \end{bmatrix}$ . The point

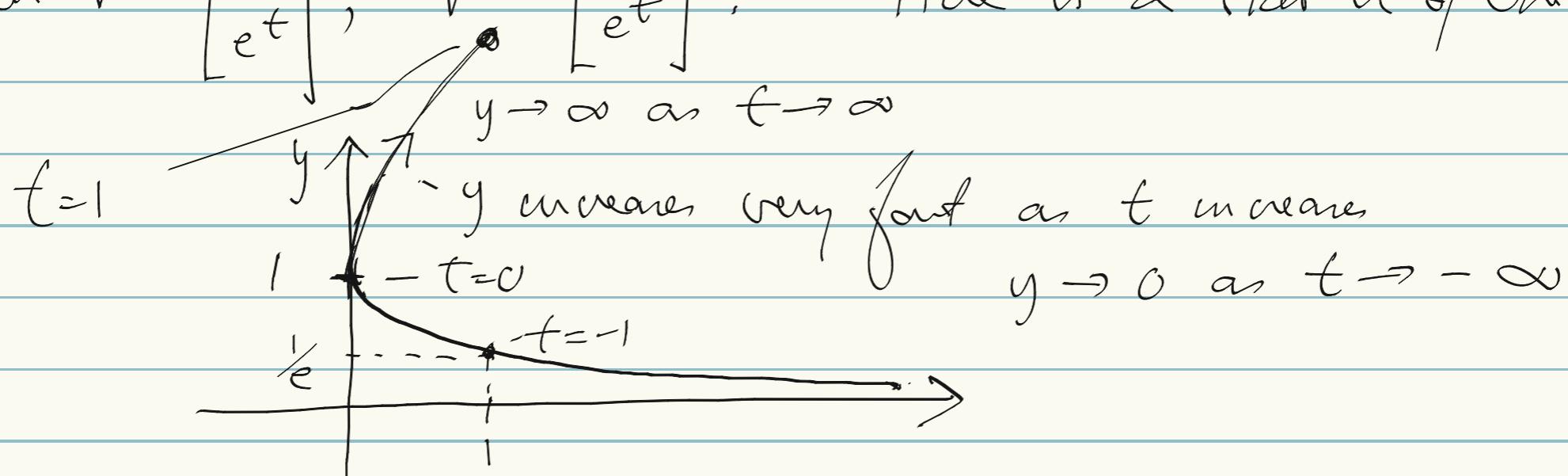
$(0, 0, 0)$  is the position at  $t=0$ , so at this point

$\vec{r}' = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ , and the tangent line is given by the

following unit-speed parametric equation:

$$\vec{r} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{t}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{t}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

15. For  $\vec{r} = \begin{bmatrix} t^2 \\ e^t \\ 1 \end{bmatrix}$ ,  $\vec{r}' = \begin{bmatrix} 2t \\ e^t \\ 0 \end{bmatrix}$ . Here is a sketch of the curve:



At  $t=0$ ,  $x=0$  and  $y=1$ ,  $\vec{r}' = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ , so the tangent

line is vertical, with equation  $x=0$ , and the speed of the particle is 1 m/s.

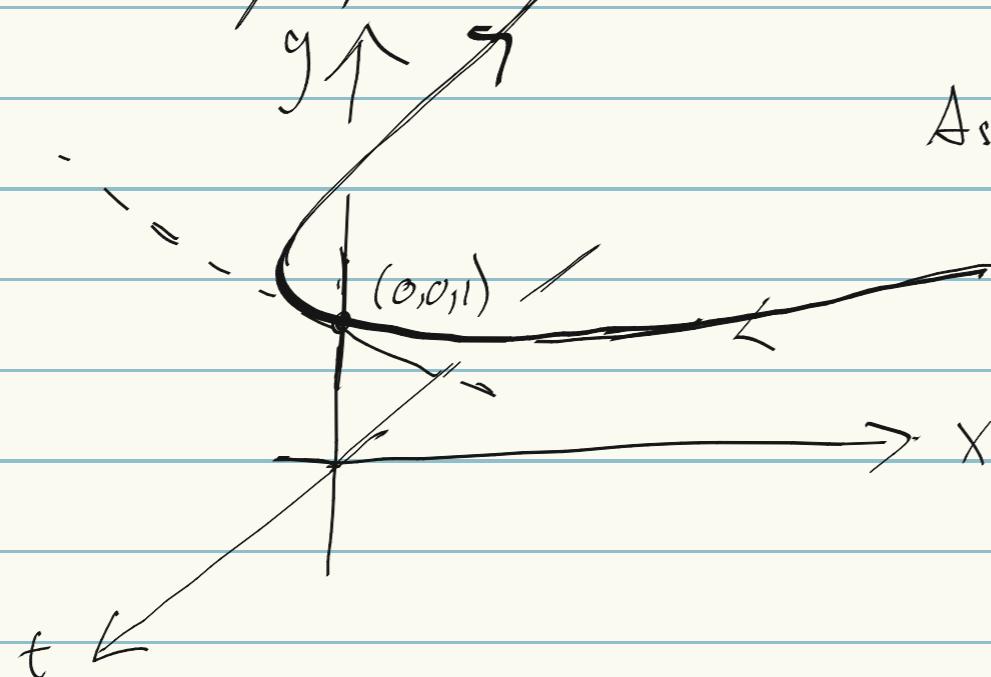
At  $t=-1$ ,  $(x, y) = (1, \frac{1}{e})$ ,  $\vec{r}' = \begin{bmatrix} -2 \\ \frac{1}{e} \\ 0 \end{bmatrix}$ . The speed at this instant is  $\sqrt{4+e^{-2}}$ . The slope of the tangent line is  $-\frac{1}{2e}$ , so the equation is  $y - \frac{1}{e} = -\frac{1}{2e}(x - 1)$ .

At  $t=1$ ,  $(x, y) = (1, e)$ ,  $\vec{r}' = \begin{bmatrix} 2 \\ e \\ 0 \end{bmatrix}$ . Now the slope is positive and steeper than at  $t=-1$ , as you can see on the graph. The speed at this instant is  $\sqrt{4+e^2}$ , and the tangent line

has equation  $y - e = \frac{e}{2}(x - 1)$ .

Again, for the graph we need three axes:  $t$ ,  $x$ , and  $y$ . I'll draw the graph with  $t$  as the independent variable, coming forward. A tangent line to this graph is analogous to the tangent line to the graph of a function  $\mathbb{R} \rightarrow \mathbb{R}$ , like you drew in Calc I.

The graph looks something like this:



As  $t$  increases,  $y$  rises exponentially as  $x$  moves in and out parabolically. The graph crosses the  $y$ -axis at  $(0,0,1)$

A parametric equation for the graph is

$$\vec{r} = \begin{bmatrix} t \\ x \\ y \end{bmatrix} = \begin{bmatrix} t \\ t^2 \\ e^t \end{bmatrix}. \quad \vec{r}' = \begin{bmatrix} 1 \\ 2t \\ e^t \end{bmatrix}.$$

At  $t=0$ , the values of  $x$  and  $y$  are 0 and 1, resp.

$\vec{r}' = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ , as in the previous example (coincidentally).

The tangent line to the graph at  $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$  has equation

$$\vec{r} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{t}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ at unit speed. It lies}$$

in the  $t, y$ -plane with slope 1. ( $\Delta y = \Delta t$ ).

For practice, find the tangent line to this graph at  $t = \pm 1$ .

