Calculus of Vector-valued Functions of a Real Variable

Charles Delman

February 9, 2014

Exercises: Evaluate the Limits

1. What is $\lim_{x \to 0} \frac{\sin x}{x}$?
2. What is $\lim_{x \to 0} x^2$?
3. What is $\lim_{x \to 0} \frac{x^3}{x}$?
4. What is $\lim_{x \to 0} 10x^2$?
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   . . .

9. Is $\lim_{x \to 0} x \sin(\frac{1}{x}) = 0$? Yes!

Exercises: Evaluate Each Limit at Infinity

1. $\lim_{x\to\infty} \frac{1}{x}$
2. $\lim_{x\to-\infty} \frac{1}{x}$
3. $\lim_{x\to\infty} \frac{1+x}{x}$
4. $\lim_{x\to\infty} \frac{\sqrt{x^2}}{x}$
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Exercises: Evaluate Each Limit (If It Exists)

1. $\lim_{x\to\infty} \frac{1}{x} = 0$
2. $\lim_{x\to-\infty} \frac{1}{x} = 0$
3. $\lim_{x\to\infty} \frac{1+x}{x} = 1$
4. $\lim_{x\to\infty} \frac{\sqrt{x^2}}{x} = 1$
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Exercises: Evaluate Each Limit (If It Exists)

6. $\lim_{x \to \infty} \frac{\sin x}{x}$
7. $\lim_{x \to -\infty} \frac{\sin x}{x}$
8. $\lim_{x \to \infty} \sin\left(\frac{1}{x}\right)$
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Exercises: Evaluate Each Limit (If It Exists)

6. $\lim_{x \to \infty} \frac{\sin x}{x} = 0$

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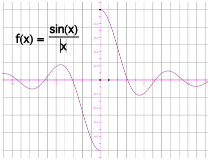
Informal Definitions of Some Types of Limits

- The limit of a *sequence*, which is just a function of positive whole numbers: $\lim_{n\to\infty} f(n) = L$ if (and only if) the output value $f(n)$ stays *arbitrarily* close to L as long as n is *sufficiently* large.
- Example: $\displaystyle\lim_{n\to\infty} \frac{n+1}{n} = 1$
- The limit of a function of a real variable as its input approaches a specified value: $\lim_{x\to a} f(x) = L$ if (and only if) the output value $f(x)$ stays *arbitrarily* close to L as long as x is *sufficiently* close to a.

Exercise: Evaluate the Limit, If It Exists

$$\lim_{x \to 0} \frac{\sin x}{|x|}$$

Exercise: Evaluate the Limit, If It Exists



It does not exist. But we can consider the weaker notion of a limit as the input approaches from the left (below) or right (above). These do exist:

$$\lim_{x \to 0^-} \frac{\sin x}{|x|} = -1$$

$$\lim_{x \to 0^+} \frac{\sin x}{|x|} = 1$$

Informal Definitions of Left and Right Limits

- The limit of a function of a real variable as its input approaches a specified value from the left (below): $\lim_{x \to a^-} f(x) = L$ if (and only if) the output value $f(x)$ stays *arbitrarily* close to L as long as x is *sufficiently* close to a and also *less than* a.
- The limit of a function of a real variable as its input approaches a specified value from the right (above): $\lim_{x \to a^+} f(x) = L$ if (and only if) the output value $f(x)$ stays *arbitrarily* close to L as long as x is *sufficiently* close to a and also *greater than* a.

Formal & Precise Definition: Finite Limit at a Finite Value

- **Definition.** $\lim_{x \to a} f(x) = L$ if (and only if), given any positive real number ϵ, there is a positive real number δ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

- **Remark:** The condition that $0 < |x - a| < \delta$ means that the value of x is within δ of a, *but not equal to* a. The limit at a requires nothing when the value of x is equal to a, where the value of $f(x)$ may be undefined.
- **Remark:** The consequence that $|f(x) - L| < \epsilon$ means that the value of $f(x)$ is within ϵ of the limiting value L. It is does not matter whether or not $f(x)$ is equal to L for some values of x satisfying the condition, hence there is no requirement that $0 < |f(x) - L|$.

Illustrative Contrasting Examples

- For example, if the function f is a constant function defined by $f(x) = c$, $c \in \mathbb{R}$, and if a is any real number, $\lim_{x \to a} f(x) = c$ because $f(x) = c$, and hence $|f(x) - c| = |c - c| = 0 < \epsilon$, for *every* value of x.
- On the other hand, if the function g is defined by $g(x) = \frac{x^2}{x}$, then $\lim_{x \to 0} g(x) = 0$, even though $g(x)$ is not equal to 0 for *any* value of x. Note that $g(x)$ is not defined for $x = 0$; 0 is not in the domain of g.
- These examples illustrate the importance of attention to details in a precise definition.

Formal & Precise Definition: Finite Limits from the Left & Right

- **Definition.** $\lim_{x \to a^-} f(x) = L$ if (and only if), given any positive real number ϵ, there is a positive real number δ such that

$$a - \delta < x < a \Rightarrow |f(x) - L| < \epsilon.$$

Exercise:

- Provide a precise, formal definition: $\lim_{x \to a^+} f(x) = L$ if (and only if) . . ..

Formal & Precise Definition: Limits at Infinity

- Just as x being *sufficiently close to, but not equal to,* a means that $0 < |x - a| < \delta$, where δ a sufficiently small positive real number, x being *sufficiently close to* $+\infty$ means that $x > N$, for some sufficiently *large* positive real number N. (Obviously, x will never equal $+\infty$.)
- It is customary to take N to be a natural number.
- Thus we make the definition of a finite limit at $+\infty$ formal as follows:

  Definition. $\lim\limits_{x \to +\infty} f(x) = L$ if (and only if), given any positive real number ϵ, there is a positive integer N such that

$$x > N \Rightarrow |f(x) - L| < \epsilon.$$

Exercise: Provide a Precise & Formal Definition

Definition. $\lim_{x \to -\infty} f(x) = L$ if (and only if) ...

Theorem: The Limit of a Sum is the Sum of the Limits

Theorem

If $\lim_{x \to a} f(x)$ *and* $\lim_{x \to a} g(x)$ *exist and are finite, then* $\lim_{x \to a} f(x) + g(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$.

- Applying the Theorem:
  - Example: $\lim\limits_{x \to 0} \dfrac{\sin x}{x} + x^2 = 1 + 0 = 1$.
  - Example: The theorem does not apply to $\lim_{x \to 0} \frac{\sin x}{x} + \frac{1}{x}$, since $\lim_{x \to 0} \frac{1}{x}$ does not exist.
  - Example: The theorem does not apply to $\lim_{x \to 0} \frac{\sin x}{x} + \frac{1}{x^2}$, since $\lim_{x \to 0} \frac{1}{x^2} = \infty$ is not finite.

How to Show a Theorem is True

- The consequence of the theorem need only hold for instances that satisfy the condition.
- If the condition is false, there is nothing to show!
- Therefore, to show that the theorem is true, we assume the condition is true; under this assumption, we must logically demonstrate the truth of the consequence.
- Please note that this assumption is *provisional*; the condition is certainly not true in all instances!
- Please also note that we must take care to *assume nothing beyond* the stated condition.

Restating the Theorem Often Helps

- Some labels make both the condition and the consequence easier to state and work with:
  - Let $\lim_{x \to a} f(x) = L$.
  - Let $\lim_{x \to a} g(x) = M$.
- Substituting these labels, we have the following restatement of the theorem:

Theorem

If $\lim_{x \to a} f(x) = L$ *and* $\lim_{x \to a} g(x) = M$*, then* $\lim_{x \to a} f(x) + g(x) = L + M$*.*

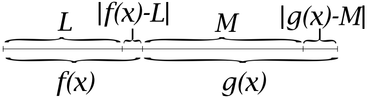
Using Definitions to Work with the Condition

- The condition that $\lim_{x \to a} f(x) = L$ means that we can make $|f(x) - L|$ as small as we like, as long as x is sufficiently close to a; *sufficiently close* means $0 < |x - a| < \delta$, for a suitable positive real number of δ.
- Similarly, we can make $|g(x) - M|$ as small as we like.
- Key point: for the smaller value of δ, *both* $|f(x) - L|$ and $|g(x) - M|$ will be as small as we like.
- So . . . how small do we need them to be?

Using Definitions to Work with the Consequence

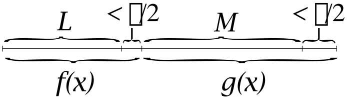
- The consequence that $\lim_{x \to a} f(x) + g(x) = L + M$ means, given any positive real number ϵ, there is a positive real number δ such that $0 < |x - a| < \delta$ is sufficient to ensure that $|f(x) + g(x) - (L + M)| < \epsilon$ (that is, $0 < |x - a| < \delta \Rightarrow |f(x) + g(x) - (L + M)| < \epsilon$).
- To show this is true, we must consider an arbitrary positive real number ϵ and show that a suitable δ exists for that ϵ.
- We will find a suitable δ by making $|f(x) - L|$ and $|g(x) - M|$ small enough to ensure that $|f(x) + g(x) - (L + M)| < \epsilon$.

A Picture Shows Why the Theorem is True



- How small must $|f(x) - L|$ and $|g(x) - M|$ be to ensure that $|f(x) + g(x) - (L + M)| < \epsilon$?

Conclusion of the Proof!



- If $|f(x) - L| < \frac{\epsilon}{2}$ and $|g(x) - M| < \frac{\epsilon}{2}$, then $|f(x) + g(x) - (L + M)| = |f(x) - L + g(x) - M| \leq |f(x) - L| + |g(x) - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Theorem: The Limit of a Product is the Product of the Limits

Theorem

If $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *exist and are finite, then* $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$.

- Applying the Theorem:
  - Example: $\lim_{x\to 0} \left(\frac{\sin x}{x}\right)\left(\frac{x^2+2x}{x}\right) = (1)(2) = 2.$
- Letting $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, we again have a restatement in a form that is easier to prove:

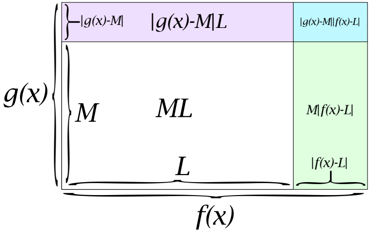
Theorem

If $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$, *then* $\lim_{x\to a} f(x)g(x) = LM$.

Using Definitions to With the Condition and the Consequence

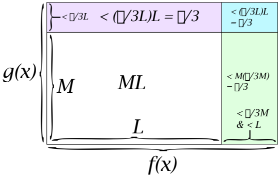
- Again, our condition tells us that we can make $|f(x) - L|$ and $|g(x) - M|$ as small as we like, as long as $0 < |x - a| < \delta$, for a suitable positive real number of δ in each case.
- Again, a simple but key observation is that a single choice of δ will work in both cases.
- Again, we must consider an arbitrary positive real number ϵ. To prove the current theorem, we must show that a suitable δ exists to ensure that $|f(x)g(x) - LM| < \epsilon$..
- Again will find a suitable δ by making $|f(x) - L|$ and $|g(x) - M|$ small enough to ensure that $|f(x)g(x) - LM| < \epsilon$.
- So ... how small do we need them to be?

A Picture Shows Why the Theorem is True



- The picture shows that $|f(x)g(x) - LM| \leq |g(x) - M||L| + |g(x) - M||f(x) - L| + |M||f(x) - L|$.

The Conclusion of the Proof!



- Choose δ such that $0 < |x - a| < \delta \Rightarrow$:
  - $|g(x) - M| < \frac{\epsilon}{3|L|}$, *and*
  - $|f(x) - L| < |L|$, *and*
  - $|f(x) - L| < \frac{\epsilon}{3|M|}$.
- Then $|f(x)g(x) - LM| < \frac{\epsilon}{3|L|} \cdot |L| + \frac{\epsilon}{3|L|} \cdot |L| + |M| \cdot \frac{\epsilon}{3|M|} = \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$.

The Limit of a Constant Function

Theorem

If c is any real number, $\lim_{x \longrightarrow a} c = c$.

This rather obvious fact follows directly from the definition of limit:

Proof.

Given any positive real number ϵ, let $\delta = 1$, or anything else you like! $0 < |x - a| < 1 \Rightarrow |c - c| = 0 < \epsilon$. $\square$

All That is Needed to Define Limits is a Notion of Distance

- For real numbers x and a, $0 < |x - a| < \delta$ just says, "The distance between x and L is less than δ and greater than 0 (hence $x \neq a$)."
- For real numbers $f(x)$ and L, $|f(x) - L| < \epsilon$ just says, "The distance between x and L is less than ϵ."
- All of the previous definitions and theorems for finite limits generalize immediately to vector-valued functions (including those with vector inputs), except that we can only multiply and divide by scalars. (In higher dimensions, you can "go to infinity" in infinitely many ways.)
- All that is required is to replace absolute value with the more general concept of the magnitude of a vector.

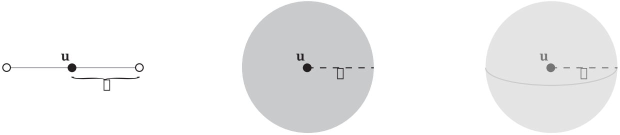
Exercise

- Decide which of the definitions and theorems on limits of real-valued functions extend in some form to vector-valued functions.
- Formulate general versions of these definitions and theorems.
- Make the necessary substitutions in the proofs of the theorems.
- Recall the definition of continuity and extend it to vector-valued functions.

Next let's use visualization to understand how limits work in higher dimensions.

Visualizing $B_\epsilon(\mathbf{u}) = \{\mathbf{v} \in \mathbb{R}^n : |\mathbf{v} - \mathbf{u}| < \epsilon\}$

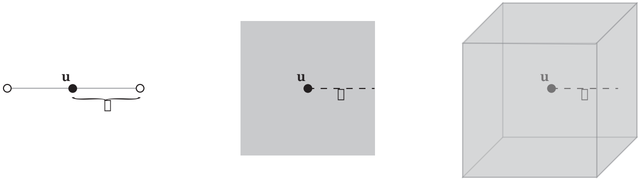
- $n = 1$: $B_\epsilon(\mathbf{u})$ is the open interval of radius ϵ centered at $\mathbf{u}$.
- $n = 2$: $B_\epsilon(\mathbf{u})$ is the open disk of radius ϵ centered at $\mathbf{u}$.
- $n = 3$: $B_\epsilon(\mathbf{u})$ is the open *ball* of radius ϵ centered at $\mathbf{u}$.



n=1 n=2 n=3

Visualizing $C_\epsilon(\mathbf{u}) = \{\mathbf{v} \in \mathbb{R}^n : |v_i - u_i| < \epsilon\}, i = 1, \ldots, n\}$

- $n = 1$: $C_\epsilon(\mathbf{u})$ is the open interval of radius ϵ centered at $\mathbf{u}$.
- $n = 2$: $B_\epsilon(\mathbf{u})$ is the open square of radius ϵ centered at $\mathbf{u}$, where the radius means the distance from the center to a side.
- $n = 3$: $B_\epsilon(\mathbf{u})$ is the open *cube* of radius ϵ centered at $\mathbf{u}$, where radius means the distance from the center to a face.

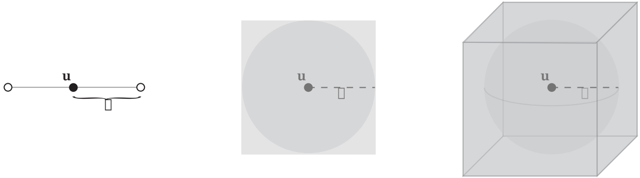


n=1 n=2 n=3

$B_\epsilon(\mathbf{u}) \subset C_\epsilon(\mathbf{u})$

$$\sqrt{\sum_{i=1}^{n}(v_i - u_i)^2} < \epsilon \Rightarrow$$

$$|v_i - u_i| = \sqrt{(v_i - u_i)^2} \leq \sqrt{\sum_{i=1}^{n}(v_i - u_i)^2} < \epsilon$$

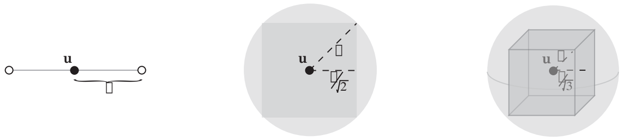


n=1 n=2 n=3

$C_{\frac{\epsilon}{\sqrt{n}}}(\mathbf{u}) \subset B_\epsilon(\mathbf{u})$

$$|v_i - u_i| < \frac{\epsilon}{\sqrt{n}}, i = 1, \ldots, n \Rightarrow$$

$$\sqrt{\sum_{i=1}^{n}(v_i - u_i)^2} < \sqrt{\sum_{i=1}^{n}(\frac{\epsilon}{\sqrt{n}})^2} = \sqrt{\sum_{i=1}^{n}\frac{\epsilon^2}{n}} = \sqrt{\epsilon^2} = \epsilon$$



n=1 n=2 n=3

It Follows that Limits May be Computed Coordinate by Coordinate

Theorem

$$\lim_{t \to a} \mathbf{v}(t) = \left(\lim_{t \to a} v_1(t), \ldots, \lim_{t \to a} v_n(t) \right)$$

Proof.

(Idea; details will be worked out in class.) If we can make $\mathbf{v}(t)$ arbitrarily close to a limiting vector when t is sufficiently close to (but not equal to) a, then we can make each coordinate just as close. (The ball is inside the cube.) Conversely, if we can make each coordinate $v_i(t)$ arbitrarily close to a limit, then we can make $\mathbf{v}(t)$ close, too. (A smaller cube, with radius shrunk by the factor $\frac{1}{\sqrt{n}}$, is inside the ball.) □

Analogous results apply to sequential limits and limits at infinity.

Exercises

1. Define what it means for a vector-valued function $\mathbf{f}$ to be continuous at a.
2. Prove that $\lim\limits_{t \to a} \mathbf{v} \cdot \mathbf{w} = \lim\limits_{t \to a} \mathbf{v} \cdot \lim\limits_{t \to a} \mathbf{w}$.
3. Prove that $\lim\limits_{t \to a} \mathbf{v} \times \mathbf{w} = \lim\limits_{t \to a} \mathbf{v} \times \lim\limits_{t \to a} \mathbf{w}$.

The Definitions of the Derivative and Riemann Integral Extend in the Obvious Way

- Let $\mathbf{v} = \mathbf{f}(t)$, a vector-valued function of t.
- All that is required for the following definitions is addition and scalar multiplication:

$$\mathbf{v}' = \mathbf{f}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{f}(t + \Delta t) - \mathbf{f}(t)}{\Delta t}$$

(if this limit exists, in which case $\mathbf{f}$ is called *differentiable*).

$$\int_a^b \mathbf{f}(t)dt = \lim_{n \to \infty} \sum_{i=1}^{n} \mathbf{f}(t_i^*)\Delta t$$

(if this limit exists), where $a = t_0, t_1, t_2, \ldots, t_n = b$ is a partition of the interval $[a, b]$, $\Delta t = \frac{b-a}{n}$, the width of each subinterval, and $t_i^* \in [t_{i-1}, t_i]$ (in which case $\mathbf{f}$ is called *Riemann integrable*).

Exercises

1. Prove the sum and scalar multiplication rules for differentiation of vector-valued functions.
2. Prove the sum and scalar multiplication rules for integration of vector-valued functions.

The results above show that differentiation and integration are *linear operators*.

3. Prove the rule for exchanging the limits of integration.

Let $\mathbf{f}(t) = \langle f_1(t), \ldots, f_n(t) \rangle$

4. Prove that $\mathbf{f}'(t) = \langle f_1'(t), \ldots, f_n'(t) \rangle$.
5. Prove that $\int_a^b \mathbf{f}(t)dt = \langle \int_a^b \mathbf{f_1}(t)dt, \ldots, \int_a^b \mathbf{f_n}(t)dt \rangle$.

On account of the results above, the Mean Value Theorem & Fundamental Theorem of Calculus apply to vector-valued functions.

The General Product Rule

- Any bilinear product will satisfy a product rule analogous to $(fg)' = f'g + fg'$.
- To see this, first recall the proof of the product rule: Let $y = f(x)$, $z = g(x)$, $y + \Delta y = f(x + \Delta x)$, $z + \Delta z = g(x + \Delta x)$. Thus $\Delta(yz) = (y + \Delta y)(z + \Delta z) - yz = (\Delta y)z + y\Delta z + \Delta y \Delta z$, and we have

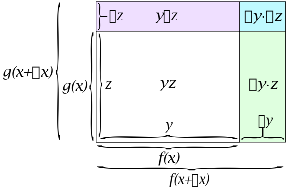
$$(yz)' = \lim_{\Delta x \to 0} \frac{\Delta(yz)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(\Delta y)z + y\Delta z + \Delta y \Delta z}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(\Delta y)}{\Delta x} z + y \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \Delta z$$

$$= y'z + yz' + (y')(0) = y'z + yz'$$

The General Product Rule (Continued)

- This proof depends only on the bilinearity of ordinary multiplication, which is just the distributive property: $a(b+c) = ab + ac$, $(a+b)c = ac + bc$.
- This picture says it all!



- Now you try! Prove that $(\mathbf{u} \cdot \mathbf{v})' = \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}'$.
- Note that we don't need commutativity, so the proof will work for the cross product as well.
- Prove that $(\mathbf{u} \times \mathbf{v})' = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}'$