4.3 Minimizing & Mixed Constraints

So far, you have seen how to solve one type of problem:

Standard Maximum

1. The objective function is to be **maximized**.

2. Constraints use $\leq$.

3. The constants in each constraint are nonnegative.

4. There are nonnegative conditions on each variable.

If a problem isn't in this form, we try to convert it to a standard maximum problem. There are a few ways that a problem can differ from the standard maximum problem. There are techniques for dealing with each of them:

4.3.1 Minimizing a function

Convert a problem involving minimization of M into a maximization problem by defining $N = -M$ and proceeding to maximize N. When N is its maximum, $M = -N$ will be at its smallest.

x	y	M	N
1	1	3	-3
1	2	4	-4
2	1	5	-5
2	2	6	-6

$(M = 2x + y$ and $N = -z = -2x - y)$

Example: Minimize: $M = 15x + 11y$

Set $N = -M = -15x - 11y$ and maximize using simplex method.

If the max value of $N = 12$, then M's minimum value is is -12.

4.3.2 $\geq$ Constraints

Convert constraints using $\geq$ to use $\leq$ by multiplying both sides by -1. Although this results in a problem that is not a standard maximum, the negative constants in the constraints will be dealt with within the simplex method.

Example: maximize:
$$z = 15x + 11y$$

subject to:

$$\begin{aligned}
2x + y &\leq 22 \\
9x + 7y &\geq 19 \\
x, y &\geq 0
\end{aligned}$$

Replace the second constraint with

$$-9x - 7y \leq -19$$

4.3.3 = Constraints

Convert constraints using $=$ into two using $\leq$ and $\geq$.

Example: maximize:
$$z = 15x + 11y$$

subject to:

$$
\begin{aligned}
2x + y &\leq 22 \\
9x + 7y &= 19 \\
x, y &\geq 0
\end{aligned}
$$

Replace the second constraint with

$$-9x - 7y \leq -19$$

$$-9x - 7y \geq -19$$

Multiply the constraint using $\geq$ by -1 on both sides:

$$-9x - 7y \leq -19$$

$$9x + 7y \leq 19$$

4.3.4 Negative Constant in ≤ Constraints - Phase 1 and 2

Consider a problem in standard maximum form, except with a negative constant in a constraint:

Example: maximize: $M = 8x + 5y$

$$\text{subject to:} \begin{cases} x + y & \le & 8 \to (0,8),(8,0) \\ 5x + 3y & \ge & 21 \to (0,7),(4.2,0) \\ x + 3y & \ge & 9 \to (0,3),(9,0) \\ x, y & \ge & 0 \end{cases}$$

$$\begin{array}{cccccc|c} x & y & u & v & w & M & \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 8 \\ -5 & -3 & 0 & 1 & 0 & 0 & -21 \\ -1 & -3 & 0 & 0 & 1 & 0 & -9 \\ \hline -8 & -5 & 0 & 0 & 0 & 1 & 0 \end{array}$$

The initial basic solution is: $x = 0, y = 0, u = 8, v = -21, w = -9, M = 0$ which is not feasible.

The standard simplex method moves from one feasible basic solution (a vertex of the feasible region) to another, so before we can use it, the tableau must be pivoted one or more times to obtain a basic **feasible** solution. This is called **Phase 1** of the problem. I will use a easier method for phase 1 than what is used in the textbook. The approach in the textbook is likely better for real world problems, and writing computer programs, but is more work when pivoting by hand.

Whatever path we take, **Phase 1** consists of **pivoting until all the slack variables are non-negative** (positive constants above the line) which means the basic solution is feasible and is a vertex of the feasible region. When there are **no negative constants**, Phase 1 is complete. In general, both Phase 1 and 2 could take several pivots each.

Phase 1 - Mattingly: Pivot on any negative coefficient in a row with a negative constant This will automatically convert the negative coefficient to a 1, and make the negative constant positive. Repeat until no negative constants remain. (slack variables nonnegative)

We can also pivot in a direction that moves closer to the optimal solution: We can see that pivoting on any of the 4 negative coefficients will get rid of at least one negative constant, but maybe some are better choices than others. If we pivot in x's column, the objective function will increase more, since its objective coefficient is **more negative** than y's. Also, if we pivot on the -5, the second row constant will be 4.25, too small to make the -9 in row 3 positive, however, if we pivot on the -1, we can fix both negative constants at once. An additional benefit is the lack of fractions when pivoting manually.

Trying to take care of multiple negative constants in one pivot and avoiding fractions may help when working problems like these by hand, but if using technology, you may pivot on negative entries "at will", although Goldstein's test ratio method given below may prevent potential loops in the simplex algorithm.

Phase 1:

$$\begin{array}{cccccc} x & y & u & v & w & M \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 8 \\ -5 & -3 & 0 & 1 & 0 & 0 & -21 \\ -1 & -3 & 0 & 0 & 1 & 0 & -9 \\ \hline -8 & -5 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Ordinarily, I'd pivot on (3,1) - do you see why?

For a more revealing example, let's pivot on (3,2):

$$\begin{array}{cccccc} x & y & u & v & w & M \end{array}$$

$$\left[\begin{array}{cccccc|c} \frac{2}{3} & 0 & 1 & 0 & \frac{1}{3} & 0 & 5 \\ -4 & 0 & 0 & 1 & -1 & 0 & -12 \\ \frac{1}{3} & 1 & 0 & 0 & -\frac{1}{3} & 0 & 3 \\ \hline -\frac{19}{3} & 0 & 0 & 0 & -\frac{5}{3} & 1 & 15 \end{array}\right]$$

$x = 0, y = 3, u = 5, v = -12, w = 0, M = 15$

Unfortunately, this made row 2's constant negative, so Phase 1 requires more pivots:

Pivoting on (2,1):

$$\begin{array}{cccccc} x & y & u & v & w & M \end{array}$$

$$\left[\begin{array}{cccccc|c} 0 & 0 & 1 & \frac{1}{6} & \frac{1}{6} & 0 & 3 \\ 1 & 0 & 0 & -\frac{1}{4} & \frac{1}{4} & 0 & 3 \\ 0 & 1 & 0 & \frac{1}{12} & -\frac{5}{12} & 0 & 2 \\ \hline 0 & 0 & 0 & -\frac{19}{12} & -\frac{1}{12} & 1 & 34 \end{array}\right]$$

$x = 3, y = 2, u = 3, v = 0, w = 0, M = 34$

This tableau is feasible, so Phase 1 is complete. We can tell it's feasible from the basic solution, but more easily by noticing that the constants are all nonnegative. We now switch to the standard simplex algorithm to choose the remaining pivot elements, and refer to the rest of the process as Phase 2.
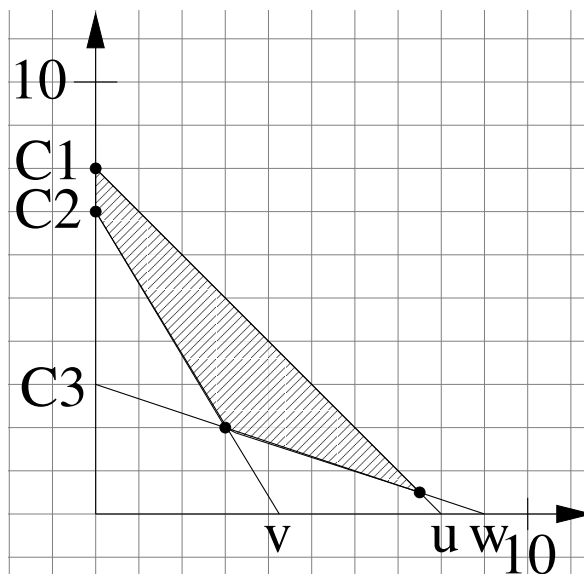
Phase 2:

Pivot on (1,4) - verify the test ratios.

$$\begin{array}{cccccc} x & y & u & v & w & M \end{array}$$

$$\left[\begin{array}{cccccc|c} 0 & 0 & 6 & 1 & 1 & 0 & 18 \\ 1 & 0 & \frac{3}{2} & 0 & \frac{1}{2} & 0 & \frac{15}{2} \\ \hline 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{19}{2} & 0 & \frac{3}{2} & 1 & \frac{125}{2} \end{array}\right]$$

$x = \frac{15}{2}, y = \frac{1}{2}, u = 0, v = 18, w = 0, M = \frac{125}{2}$

This tableau is optimal, so Phase 2 is complete. M's maximum is $\frac{125}{2}$ when $x = \frac{15}{2}, y = \frac{1}{2}$

Note that each phase could take several pivots. You should try to solve this problem yourself using technology, picking a few different first phase 1 pivots, and seeing how things play out.

Phase 1 - Goldstein - not recommended when working by hand: Find a **nonfeasible row** with both a negative constant and at least one negative coefficient. Select a **pivot column** with a negative coefficient in this nonfeasible row, and then compute test ratios to determine the pivot element as usual. Repeat until no negative entries in the last column remain. (slack variables nonnegative)
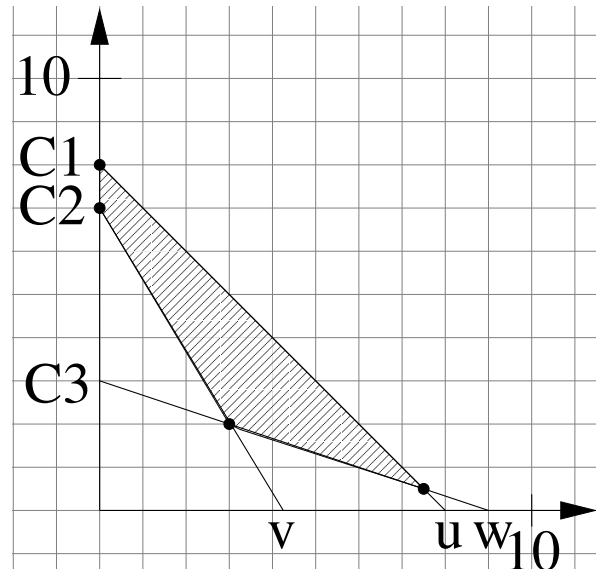
$$
\begin{array}{cccccc|c}
x & y & u & v & w & M & \\
1 & 1 & 1 & 0 & 0 & 0 & 8 \\
-5 & -3 & 0 & 1 & 0 & 0 & -21 \\
-1 & -3 & 0 & 0 & 1 & 0 & -9 \\
-8 & -5 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

Pivoting in y's column yields:

$$
\begin{array}{cccccc|c}
x & y & u & v & w & M & \\
\frac{2}{3} & 0 & 1 & 0 & \frac{1}{3} & 0 & 5 \\
-4 & 0 & 0 & 1 & -1 & 0 & \boxed{-12} \\
\frac{1}{3} & 1 & 0 & 0 & -\frac{1}{3} & 0 & 3 \\
-\frac{19}{3} & 0 & 0 & 0 & -\frac{5}{3} & 1 & 15
\end{array}
$$

$x = 0, y = 3, u = 5, \boxed{v=-12}, w = 0, M = 15$

which is not feasible, and so we continue Phase 1:

There are two potential pivot columns.
Pivoting on the -1 in (2,5) yields:

$$
\begin{array}{cccccc|c}
x & y & u & v & w & M & \\
-\frac{2}{3} & 0 & 1 & \frac{1}{3} & 0 & 0 & 1 \\
4 & 0 & 0 & -1 & 1 & 0 & 12 \\
\frac{5}{3} & 1 & 0 & -\frac{1}{3} & 0 & 0 & 7 \\
\frac{1}{3} & 0 & 0 & -\frac{5}{3} & 0 & 1 & 35
\end{array}
$$

$x = 0, y = 7, \boxed{u=1,v=0,w=12}, M = 35$

which is feasible, so Phase 1 is complete.

Phase 2: Use the standard simplex method to find the optimal solution:

Pivoting on (1,4) yields:

$$
\begin{array}{cccccc|c}
x & y & u & v & w & M & \\
-2 & 0 & 3 & 1 & 0 & 0 & 3 \\
2 & 0 & 3 & 0 & 1 & 0 & 15 \\
1 & 1 & 1 & 0 & 0 & 0 & 8 \\
-3 & 0 & 5 & 0 & 0 & 1 & 40
\end{array}
$$

$x = 0, y = 8, \boxed{u=0, v=3, w=15}, M = 40$

Pivoting on (2,1) yields the final tableau:

$$
\begin{array}{cccccc|c}
x & y & u & v & w & M & \\
0 & 0 & 6 & 1 & 1 & 0 & 18 \\
1 & 0 & \frac{3}{2} & 0 & \frac{1}{2} & 0 & \frac{15}{2} \\
0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\
\hline
0 & 0 & \frac{19}{2} & 0 & \frac{3}{2} & 1 & \frac{125}{2}
\end{array}
$$

$x = \frac{15}{2}, y = \frac{1}{2}, u = 0, v = 18, w = 0, M = \frac{125}{2}$

M's maximum is $\frac{125}{2}$ when $x = \frac{15}{2}, y = \frac{1}{2}$

This approach to phase 1 is pretty inefficient in this problem. The simplex algorithm assumes we are on the feasible region, in which case the test ratios assure we pivot to the nearest intersection with a constraint, which is a vertex of the feasible region. If we're not inside the feasible region, the test ratio method unnecessarily requires more pivots.