

4 Linear Programming: The Simplex Method

4.1 Setting Up the Simplex Method

We will now study a technique that allows us to solve more complex linear programming problems. This technique converts the constraints to a system of linear equations, so we can use matrix techniques to solve the system. This is a specific technique that applies only to linear programming problems that have been put in **standard maximum form**.

4.1.1 Standard Maximum

1. The objective function is to be maximized.
2. Each constraint is written using \leq .
3. The constants in each constraint are nonnegative.
4. There are nonnegative conditions on each variable.

4.1.2 Slack Variables

We will now discuss the process of changing a system of linear inequalities into a system of linear equations.

Suppose we have a linear programming problem:

Maximize

$$M = 10x + 15y$$

subject to:

$$\begin{aligned} x + 5y &\leq 65 \rightarrow (0, 13), (65, 0) \\ 2x + 3y &\leq 32 \rightarrow (0, 10.66), (16, 0) \\ 4x + y &\leq 28 \rightarrow (0, 16), (16, 0) \\ x, y &\geq 0 \end{aligned}$$

Consider the first constraint: if $x + 5y \leq 65$, then there is some value $u \geq 0$ such that $x + 5y + u = 65$

For example, $(15, 5)$ satisfies this constraint: $15 + 5(5) \leq 65$ or $15 + 5(5) + 15 = 65$

So when $x = 15, y = 5, u = 15$

$(5, 15)$ doesn't satisfy this constraint: $5 + 5(15) \not\leq 65$ or $5 + 5(15) - 15 = 65$

So when $x = 5, y = 15, u = -15$

If (x, y) satisfies this constraint, $u \geq 0$, and if it doesn't, then $u < 0$.

We can represent the inequality $x + 5y \leq 65$ as an equation with a nonnegative condition on u :

$$x + 5y + u = 65, u \geq 0$$

The point (15,5) satisfies constraint 1, but not constraint 2. Since any point (x,y) may have different amounts of slack with respect to the different constraints, we must use a different slack variable for each constraint.

We express the constraints as linear equations:
$$\begin{cases} x + 5y + u & = 65 \\ 2x + 3y + v & = 32 \\ 4x + y + w & = 28 \end{cases}$$

and include the objective function in the system of equations, by moving all of the variable terms to the left side of the equation.

$$-10x - 15y + M = 0$$

$$\begin{cases} x + 5y + u & = 65 \\ 2x + 3y + v & = 32 \\ 4x + y + w & = 28 \\ -10x - 15y + M & = 0 \\ x, y, u, v, w, & \geq 0 \end{cases}$$

We usually don't include the nonnegative conditions, but take them as given.

4.1.3 furniture manufacturing problem

Example: Write as a system of equations:

Maximize $p = 80x + 70y$

subject to:
$$\begin{cases} 6x + 3y \leq 96 \\ x + y \leq 18 \\ 2x + 6y \leq 72 \\ x, y, z, x_4 \geq 0 \end{cases}$$

$$\begin{cases} 6x + 3y + u & = 96 \\ x + y + v & = 18 \\ 2x + 6y + w & = 72 \\ -80x - 70y + M & = 0 \end{cases}$$

4.1.4 The Simplex Tableau

Example: Set up the simplex tableau:

$$\begin{array}{cccccc|c} x & y & u & v & w & p & \\ \hline 6 & 3 & 1 & 0 & 0 & 0 & 96 \\ 1 & 1 & 0 & 1 & 0 & 0 & 18 \\ 2 & 6 & 0 & 0 & 1 & 0 & 72 \\ \hline -80 & -70 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Example: Work exercises from the text: 6

Homework section 4.1: 1-5 odd

4.1.5 Basic solution

Tableau columns that contain exactly one nonzero entry of 1 are called **unit columns**. The variables with unit columns in the simplex tableau are called **basic** (Group II) variables. The other variables are called **nonbasic** (Group I) variables.

In the initial tableau, the slack and objective variables are always basic. u, v , and z are the basic variables.

To obtain the **basic solution** for any tableau:

1. Solve for the basic (Group II) variables
2. Set the nonbasic (Group I) variables equal to zero

$$u = 96 - 6x - 3y = 96$$

$$v = 18 - x - y = 18$$

$$w = 72 - 2x - 6y = 72$$

We see that each of the slack variables are nonnegative, so this is a **basic feasible solution**. That means it is a solution to the original system of inequalities, as well as the system of equations.

This basic feasible solution represents the corner of the feasible region at the origin. As we progress through the Simplex method, we can examine each new tableau's basic solutions to see what corner point of the feasible region we're at, and verify that we're still in the feasible region.

In practice, we can simply ignore the nonbasic columns, and **solve for the basic variables, directly from the tableau**.

4.1.6 simplex pivot

When we pivot a simplex tableau, **we may only obtain a 1 in the pivot location by multiplying**. We'll see why this is in the next section.

Example: Work exercises from the text: 14,16

Homework section 4.1: 1-15 odd

4.2 The Simplex Method

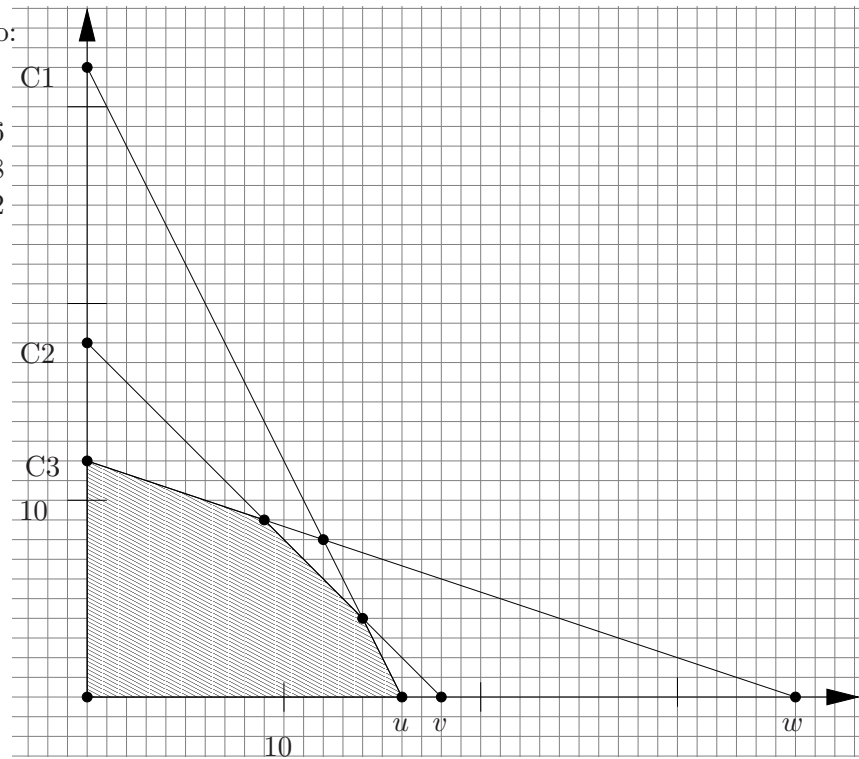
4.2.1 furniture manufacturing problem

We will use the simplex method to:

$$\text{Maximize } p = 80x + 70y$$

$$\text{subject to: } \begin{cases} 6x + 3y \leq 96 \\ x + y \leq 18 \\ 2x + 6y \leq 72 \\ x, y, z, x_4 \geq 0 \end{cases}$$

Vertex	$p = 80x + 70y$
(0,0)	0
(0,12)	840
(16,0)	1280
(9,9)	1350
(14,4)	1400



4.2.2 Simplex solution

This is a standard maximum problem which yields the tableau:

$$\left[\begin{array}{cccccc|c} x & y & u & v & w & p & \\ \hline 6 & 3 & 1 & 0 & 0 & 0 & 96 \\ 1 & 1 & 0 & 1 & 0 & 0 & 18 \\ 2 & 6 & 0 & 0 & 1 & 0 & 72 \\ \hline -80 & -70 & 0 & 0 & 0 & 1 & 0 \end{array} \right] \text{ with basic solution is } \begin{cases} x = 0 \\ y = 0 \\ u = 96 \\ v = 18 \\ w = 72 \\ p = 0 \end{cases}$$

We need to attain a basic feasible solution that makes p larger. This is done by **pivoting** on a particular element of the tableau. When we pivot, the basic feasible solution moves from one vertex to another. Specifically, we choose the pivot **column** so that the basic solution moves along an edge of the feasible region in the direction in which the objective function increases the most. We choose the pivot **row** so that the basic solution moves along this edge to the first intersection with another boundary, which is an adjacent vertex.

4.2.3 Choosing the pivot element:

1. Select the **pivot column** - the one with the **most negative coefficient** in the **bottom row**.
2. Select the **pivot row** by calculating each row's **test ratio**: Divide the rightmost entry of each row (the constant) by the pivot column's element (a coefficient) in that row. The pivot row is the one with the **smallest nonnegative test ratio**.
3. the **pivot element** is the element in the pivot column and pivot row.

The pivot element is in position (1,1)

<i>test-ratio</i>	<i>x</i>	<i>y</i>	<i>u</i>	<i>v</i>	<i>w</i>	<i>p</i>		
(16)	(6)	3	1	0	0	0	96	
18	1	1	0	1	0	0	18	
36	2	6	0	0	1	0	72	
	(-80)	-70	0	0	0	1	0	

4.2.4 Pivoting on an element

- ...in the (r, c) position consists of two steps:
1. Multiply row r by $\frac{1}{a_{rc}}$ (do not swap rows or add one row to another. If we swap rows, the pivot entry goes along, and if we add one row to the pivot row, we lose unit columns, and therefore lose the basic solution.)
 2. Zero out the rest of the pivot column c by adding multiples of pivot row r to the other rows

We will multiply R1 by $\frac{1}{6}$

<i>x</i>	<i>y</i>	<i>u</i>	<i>v</i>	<i>w</i>	<i>p</i>		
1	$\frac{1}{2}$	$\frac{1}{6}$	0	0	0	16	
1	1	0	1	0	0	18	
2	6	0	0	1	0	72	
-80	-70	0	0	0	1	0	

Add multiples of the pivot row to zero out the rest of the column:

	<i>x</i>	<i>y</i>	<i>u</i>	<i>v</i>	<i>w</i>	<i>p</i>		
$R_2 - R_1$	1	$\frac{1}{2}$	$\frac{1}{6}$	0	0	0	16	
$R_3 - 2R_1$	0	$\frac{1}{2}$	$-\frac{1}{6}$	1	0	0	2	
$R_4 + 80R_1$	0	5	$-\frac{1}{3}$	0	1	0	40	
	0	-30	$\frac{40}{3}$	0	0	1	1280	

with basic solution

$$\left\{ \begin{array}{l} x = 16 \\ y = 0 \\ u = 0 \\ v = 2 \\ w = 40 \\ p = 1280 \end{array} \right.$$

The basic solution has moved from $(0,0)$ to $(16,0)$. p has increased, but this is not the optimum solution. Since there are negative entries in the bottom row, we will select a new pivot element, and continue pivoting until there were no negative entries in the bottom row.

<i>test-ratio</i>	x	y	u	v	w	p		
32	1	$\frac{1}{2}$	$\frac{1}{6}$	0	0	0	16	
(4)	0	$(\frac{1}{2})$	$-\frac{1}{6}$	1	0	0	2	
8	0	5	$-\frac{1}{3}$	0	1	0	40	
	0	(-30)	$\frac{40}{3}$	0	0	1	1280	

pivots to

	x	y	u	v	w	p		
	1	0	$\frac{1}{3}$	-1	0	0	14	
	0	1	$-\frac{1}{3}$	2	0	0	4	
	0	0	$-\frac{4}{3}$	-10	1	0	20	
	0	0	$\frac{10}{3}$	60	0	1	1400	

with basic solution:

$$\left\{ \begin{array}{l} x = 14 \\ y = 4 \\ u = 0 \\ v = 0 \\ w = 20 \\ p = 1400 \end{array} \right.$$

The lack of negative coefficients indicates we cannot increase p further. This solution agrees with our geometric solution.

Now, let's go back to the original tableau, and see what happens if we pivot on a different element.

4.2.5 3 dimensional problem.

Example: The maximum daily production of an oil refinery is 1900 barrels. The refinery can produce 3 fuels: gasoline, diesel, and heating oil. Production costs per barrel are \$6 for gasoline, \$5 for diesel, and \$8 for heating oil. The daily production budget is \$13,400. Profit per barrel is \$8 for gasoline, \$6 for diesel, and \$9 for heating oil.

How much of each type of fuel must be produced to maximize profit?

maximize:

$$p = 8x + 6y + 9z$$

subject to:

$$\begin{aligned} x + y + z &\leq 1900 \\ 6x + 5y + 8z &\leq 13400 \end{aligned}$$

<i>test-ratio</i>	x	y	z	u	v	p		
1900	1	1	1	1	0	0	1900	
1675	6	5	(8)	0	1	0	13400	
	-8	-6	(-9)	0	0	1	0	

at (0,0,0) profit is 0

$$\begin{array}{l}
 R_1 - \frac{1}{8}R_2 \\
 \frac{1}{8}R_2 \\
 R_3 + \frac{9}{8}R_2
 \end{array}
 \left[\begin{array}{cccccc|c}
 x & y & z & u & v & p & \\
 \hline
 \frac{1}{4} & \frac{3}{8} & 0 & 1 & -\frac{1}{8} & 0 & 225 \\
 \frac{3}{4} & \frac{5}{8} & 1 & 0 & \frac{1}{8} & 0 & 1675 \\
 \hline
 -\frac{5}{4} & -\frac{3}{8} & 0 & 0 & \frac{9}{8} & 1 & 15075
 \end{array} \right]$$

at (0,0,1675) profit is \$15075

$$\begin{array}{l}
 \text{test-ratio} \\
 900 \\
 2233.\bar{3}
 \end{array}
 \left[\begin{array}{cccccc|c}
 x & y & z & u & v & p & \\
 \hline
 (\frac{1}{4}) & \frac{3}{8} & 0 & 1 & -\frac{1}{8} & 0 & 225 \\
 \frac{3}{4} & \frac{5}{8} & 1 & 0 & \frac{1}{8} & 0 & 1675 \\
 \hline
 (-\frac{5}{4}) & -\frac{3}{8} & 0 & 0 & \frac{9}{8} & 1 & 15075
 \end{array} \right]$$

$$\begin{array}{l}
 4R_1 \\
 R_2 - 3R_1 \\
 R_3 + 5R_4
 \end{array}
 \left[\begin{array}{cccccc|c}
 x & y & z & u & v & p & \\
 \hline
 1 & \frac{3}{2} & 0 & 4 & -\frac{1}{2} & 0 & 900 \\
 0 & -\frac{1}{2} & 1 & -3 & \frac{1}{2} & 0 & 1000 \\
 \hline
 0 & \frac{3}{2} & 0 & 5 & \frac{1}{2} & 1 & 16200
 \end{array} \right]$$

at (900,0,1000) profit is \$16200

Solution: Maximum profit is \$16,200 when 900 barrels of gas, 0 barrels of diesel, and 1000 barrels of heating oil are produced.

Homework section 4.2: 5-19 odd

Show 3-dim. feasible region example.