Equivalencies Used to Solve Absolute Value Equations & Inequalities

by Chad Mattingly

When solving equations and inequalities involving absolute values contining variables, we cannot isolate the variable until we get rid of the absolute value bars, so in each case, we consider what values the expression inside them could have, and write those as systems of equations or inequalities using the words "and" or "or".

1 Equations with one absolute value

Example: $|4x + 2| = 6$

First, the logic:

Which numbers have absolute value 6? (meaning they are 6 units from 0 on the RNL)

There are two: 6 and -6, so $4x + 2$ could be either value.

Therefore, the absolute value equation is equivalent to the compound equation:

$$4x + 2 = 6 \text{ or } 4x + 2 = -6$$
$$4x = 4 \text{ or } 4x = -8$$
$$x = 1 \text{ or } x = -2$$

In general, if X is any expression, and $a \geq 0$:

$|X| = a$ is equivalent to $X = -a$ or $X = a$

If $a < 0$, there is no solution, since the left side of the equation can be no less than 0.

If an equation has a single absolute value expression containing variables, isolate the absolute value expression first before applying the above equivalency. This is called putting the equation into **standard form**.

2 Equations with two absolute values

What if there are two? In how many ways could two expressions have the same absolute value, say 5?

Notice that $|5| = |5|$ $|-5| = |-5|$ $|5| = |-5|$ $|-5| = |5|$

Of these 4 cases, there are only two fundamental possibilities we need to model with equations: the two expressions must either be the **same** value or **opposite** values.

Example: $|4x - 5| = |3x + 5|$

We convert this to a compound inequality where each equation models each possibility:

$$\overbrace{4x - 5 = 3x + 5}^{\text{“same” case}} \quad \text{or} \quad \overbrace{4x - 5 = -(3x + 5)}^{\text{“opposite” case}}$$

In general, if X and $|Y|$ are any expressions, an equation of the form:

$$|X| = |Y|$$

is equivalent to

$$X = Y \text{ or } X = -Y$$

2.1 nonstandard form

Although we usually avoid tampering with absolute value expressions, in order to achieve the “standard form”, we find that we may distribute a factor outside the absolute value bars into the expression inside to enable us to apply this equivalency.

$$|X| = a|Y|$$

is equivalent to

$$|X| = |aY|$$

3 Absolute Value Inequalities with $<$ or $\leq$

In this case, the absolute value of the expression is small, a number close to 0 on the real number line:

Example: $|4x + 2| \leq 5$

First, the logic:

Which numbers have absolute values ≤ 5? (meaning they are less than 5 units from 0)

They are the interval of real numbers from -5 to 5, so $4x + 2$ could be any of these values.

Therefore, the absolute value inequality equivalent to the compound inequality:

$$-5 \leq 4x + 2 \leq 5$$

$$-7 \leq 4x \leq 3$$

$$-\frac{7}{4} \leq x \leq \frac{3}{4}$$

$$\text{with solution set: } \left[-\frac{7}{4}, \frac{3}{4}\right]$$

In general, if X is any expression, and $a \geq 0$:

$$|X| < a \text{ is equivalent to } -a < X < a$$

and

$$|X| \leq a \text{ is equivalent to } -a \leq X \leq a$$

If $a < 0$, there is no solution.

4 Absolute Value Inequalities with $>$ or $\geq$

In this case, the absolute value of the expression is large, a number far from 0 on the real number line:

Example: $|2x - 3| > 5$

First, the logic:

Which real numbers have absolute value is greater than 5? (more than 5 units from 0)

They are the interval to the left of -5, and the interval to the right of 5. $2x - 3$ could be any of these values.

Therefore, the absolute value inequality is equivalent to the compound inequality:

$$2x - 3 < -5 \text{ or } 2x - 3 > 5$$

$$2x < -2 \text{ or } 2x > 8$$

$$x < -1 \text{ or } x > 4$$

$$\text{which is } (-\infty, -1) \cup (4, \infty)$$

In general, if X is any expression:

$$|X| > a \text{ is equivalent to } X < -a \text{ or } X > a$$

and

$$|X| \geq a \text{ is equivalent to } X \leq -a \text{ or } X \leq a$$

While this equivalency is true for all real a, it's worth noticing that for $a \leq 0$ the original inequality is true for all real numbers, so working through the algebraic solution is not necessary.