Mathematics 2120 Mattingly Gauss-Jordan Supplement to Goldstein

## 2.1 Systems With Unique Solutions

There are many ways to reduce a system to diagonal form, but in this course we will use a step-by-step procedure called the **Gauss-Jordan Method** of elimination. We will focus on this technique, since it prepares you for the Simplex Method of solving linear programming problems later.

### 2.1.1 Elementary Row Operations

The process involves a series of **elementary row operations** (EROs):

1. **Swap** the locations of two rows
2. **Multiply** an row by a nonzero number (MPE)
3. Alter an row by **adding a multiple of another row** to it. (APE)

It's important to you and me that you write down the EROs you use, so that you can retrace your steps if you get the wrong answer, and so I can follow your work easily enough to award partial credit. My convention is that whenever I write a new row, I note the operations that created it on the left side of the row.

### 2.1.2 ERO notation

**Example:** $R_1 \leftrightarrow R_2$ means swap rows 1 and 2

**Example:** $4R_2$ means multiply Row 2 by 4

**Example:** $R_3 - 2R_2$ means replace row 3 with the result when (-2) times row 2 is added to row 3.

We will follow the convention that the first row mentioned in an operation is the row being changed, unless it's a swap.

### 2.1.3 referencing elements

We may refer to an element of a matrix by its location: "row $r$ and column $c$" or position $(r, c)$.

We call the elements in locations $(n, n)$ the **main diagonal**

### 2.1.4 Pivoting

To **pivot** on an element in position $(r, c)$ means to do two things:

1. Get an 1 in row $r$, column $c$, using ERO's.
2. Get 0s in the rest of column $c$ by adding multiples* of the pivot ($r^{th}$) row to them.
   (*opposite the element being zeroed)

### 2.1.5 Gauss-Jordan Method in terms of pivoting

The G-J method can now be stated: **"pivot on the main diagonal elements from left to right"**

## 2.2 Gauss-Jordan Method for General Systems of Equations

In the last section, we worked with systems of linear equations that had unique solutions. Now, we will expand our use of the Gauss-Jordan Method to include systems with infinitely many or no solutions, so the systems will not reduce to diagonal form.

Without the goal of diagonal form, we begin by describing a new goal, a standard form called **reduced row echelon form**:

A matrix is in **reduced row echelon form** if:

1. All rows consisting entirely of zeros are grouped at the bottom of the matrix.
2. The leftmost nonzero element of each row is 1. This element is called the **leading 1** of the row.
3. The leading 1 of each row is to the right of the leading 1 of the previous rows.
4. All entries above and below a leading 1 are zeros (**unit columns**).

or more concisely:

1. zero rows at bottom
2. leading 1s in each nonzero row.
3. leading 1s L→R.
4. leading 1s in unit columns.

Notice that **diagonal form** is a special case of reduced echelon form.

**Example:** Reduce the matrix to reduced row echelon form, and determine solution set:

$$\left[\begin{array}{cc|c} 2 & 6 & 4 \\ 3 & 5 & 12 \\ -2 & 0 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \text{ or } \left\{\begin{array}{rcl} x & = & 0 \\ y & = & 0 \\ 0 & = & 1 \end{array}\right.$$

so this system has **no solution**.

*(For more than 3 variables, in these notes I use $x_1, x_2, x_3, x_4$ instead of $x, y, z, ?$)*

**Example:** Reduce the matrix to reduced row echelon form, and determine solution set:

$$\left[\begin{array}{cccc|c} 1 & 3 & 6 & -2 & -7 \\ -2 & -5 & -10 & 3 & 10 \\ 1 & 2 & 4 & 0 & 0 \\ 0 & 1 & 2 & -3 & -10 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ or } \left\{\begin{array}{rcrcrcrcr} x_1 & & & & & & & = & 2 \\ & & x_2 & + & 2x_3 & & & = & -1 \\ & & & & & & x_4 & = & 3 \\ & & & & & & 0 & = & 0 \end{array}\right.$$

Now we need to be able to write this solution set in some usable form.

### 2.2.1 Writing an infinite solution set

We solve each equation for the variable corresponding to its **leading one**, and let variables which don't correspond to leading 1's in any equations take on any real value. Those variables without leading 1's are called **parameters**:

$$\left\{\begin{array}{rcr} x_1 &=& 2 \\ x_2 &=& -1-2x_3 \\ x_3 &=& \text{any real no.} \\ x_4 &=& 3 \end{array}\right\} \text{ some solutions: } \left\{\begin{array}{rcr} x_1 &=& 2 \\ x_2 &=& -1 \\ x_3 &=& 0 \\ x_4 &=& 3 \end{array}\right\}, \left\{\begin{array}{rcr} x_1 &=& 2 \\ x_2 &=& -3 \\ x_3 &=& 1 \\ x_4 &=& 3 \end{array}\right\}, \left\{\begin{array}{rcr} x_1 &=& 2 \\ x_2 &=& 13 \\ x_3 &=& -7 \\ x_4 &=& 3 \end{array}\right\}$$

In this way, we can let the independent variable $x_3$ be any real number, and then compute $x_2$ based on that value. By varying the value of the independent variable, we have access to all the solution points. In this example, sliding $x_3$ around the real number line moves our solution point along a straight line in 4-dimensional space, where $x_1$ and $x_4$ are constant, and $x_2$ and $x_3$ are related by the formula we found. Any solution set with one parameter is a straight line in 2 or more dimensions.

**Warning -** Our text says to solve for variables over columns that are in "proper form" (unit columns), which contain only a single nonzero entry of 1, and while this is *close* to being true, it's not always. In the following example, the third column *is* a unit column, but it does *not* contain a leading 1, and solving for it would lead to circular definitions of $x_2$ and $x_3$. To avoid this pitfall, remember to **solve each equation for the variable corresponding to the leading one**.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Sometimes we may solve for the variables corresponding to the leading ones, and find them to be in terms of more than one independent variable:

**Example:** $$\left[\begin{array}{cccc|c} 1 & 2 & 3 & -1 & 4 \\ 2 & 3 & 0 & 1 & -3 \\ 3 & 5 & 3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & -9 & 5 & -18 \\ 0 & 1 & 6 & -3 & 11 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ or } \left\{\begin{array}{rrrrcr} x_1 & & -\ 9x_3 & +\ 5x_4 & = & -18 \\ & x_2 & +\ 6x_3 & -\ 3x_4 & = & 11 \\ & & & 0 & = & 0 \end{array}\right.$$

$$\text{which yields: } \left\{\begin{array}{rcrl} x_1 &=& 9x_3-5x_4-18 & \\ x_2 &=& -6x_3+3x_4+11 & \\ x_3 &=& \text{any real no.} & (1^{st} \text{ parameter}) \\ x_4 &=& \text{any real no.} & (2^{nd} \text{ parameter}) \end{array}\right.$$

In this solution set, varying $x_3$ and $x_4$ moves the solution point around some 2-dimensional plane in 4-dimensional space.