Factoring Trinomials

by Chad Mattingly

1 monic trinomials ($x^2 + bx + c$)

If $x^2 + bx + c$ factors over the integers, the two factors must be of the form $(x + m)$ and $(x + n)$ for some integers m and n:

$$x^2 + bx + c = (x + m)(x + n) = x^2 + nx + mx + mn = x^2 + (m + n)x + mn$$

So to factor $x^2 + bx + c$, we only need to find m, n such that $m \cdot n = c$ and $m + n = b$. If no such m and n exist, the polynomials is prime.

We find m, n by making a list of candidates that satisfy one of these conditions, and see if any satisfy the other condition. Since we cannot list of all the m, n that have sum b, we list all m, n with product c, and check to see if anyof them add up to b:

Example: Factor $x^2 + 5x - 24$

We need $mn = -24$ and $m + n = 5$

We list all of the pairs of integers with product -24, and see if any have a sum of 5, starting with m=1, working up through all the numbers that evenly divide 24:

m	n	$m + n$	
1	-24	-23	
2	-12	-10	
3	-8	-5	
4	-6	-2	
6	-2	4	
8	-3	5	$\leftarrow$ these add up to 5, so $x^2 + bx + c = (x + 8)(x - 3)$
12	-2	10	
24	-1	23	

This is the basic idea of factoring $x^2 + bx + c$. We check pairs of numbers with product equal to c, until we find one with the sum of b.

There is no reason to continue listing potential pairs of m and n after we find the one with the desired sum, but I've listed them all to illustate two points:

First, we must check all possible values of m and n until the desired sum is achieved. If that sum is never achieved by any of the possible values, it means $x^2 + bx + c$ is **prime**, so you should be able to form the whole list to verify that a trinomial is prime.

Second, this list was longer than it really had to be. With a little thought, we can minimize the number of candidates for m and n and shorten this list by looking at the signs of c and b:

2 Optimizing our search for m and n

sign of the product:

We needed $mn = -24$, a **negative product**, so m and n must have **opposite signs**.

(If m and n had a **positive product**, they would have had the **same sign**.)

sign of the sum:

We needed $m + n = 5$, a **positive sum**, so the positive number must be "larger" in the sense of its absolute value. I signify this by writing $|+| > |-|$

In light of these discoveries, we now know we're looking for a "large" positive number and a small negative number, so our list is half as long.

m	n	$m+n$	
-1	24	23	← notice I'm starting with $m = -1$ now
-2	12	10	
-3	8	5	← we found it much sooner!
-4	6	2	

In general, there are 4 possible cases:

	positive product (same sign)	negative product (opposite signs)
positive sum	both are positive	$\|+\| > \|-\|$
negative sum	both are negative	$\|-\| > \|+\|$

3 non-monic trinomials ($ax^2 + bx + c$)

First, do not use either of these methods until you have checked to see if:

1. there is a GCF - Your work will be much harder if you don't factor out a GCF first.
2. this is a special product - perfect square trinomial or difference of squares.

If $ax^2 + bx + c$ factors over the integers, the two factors must be of the form $(px + q)$ and $(rx + s)$ for some integers $p, q, r,$ and s.

There are two methods to find these binomial factors- "Guess and Check" and "Factoring by Grouping"

3.1 Guess and Check

We try various p, r such that $pr = a$, and for each pair, we check all possible q, s such that $qs = c$ until we find $(px + q)(rx + s) = ax^2 + bx + c$

This method is easy to learn, and works well for simple trinomials, but when a and c have many factors, it becomes tiresome to list all the possibilities, especially if none work (the trinomial is prime).

Example: Factor $6x^2 - 7x - 20$ by Guess and Check.

p	r	q	s		p	r	q	s		
6	1	1	-20	$(6x + 1)(x - 20)$	3	2	1	-20	$(3x + 1)(2x - 20)$	
6	1	2	-10	$(6x + 2)(x - 10)$	3	2	2	-10	$(3x + 2)(2x - 10)$	
6	1	4	-5	$(6x + 4)(x - 5)$	3	2	4	-5	$(3x + 4)(2x - 5)$	←aha!
6	1	5	-4	$(6x + 5)(x - 4)$	3	2	5	-4	$(3x + 5)(2x - 4)$	
6	1	10	-2	$(6x + 10)(x - 2)$	3	2	10	-2	$(3x + 10)(2x - 2)$	
6	1	20	-1	$(6x + 20)(x - 1)$	3	2	20	-1	$(3x + 20)(2x - 1)$	

3.2 Factoring by Grouping

3.2.1 the procedure

To factor $ax^2 + bx + c$ by grouping:

- Find integers m, n such that $m \cdot n = ac$ and $m + n = b$.
- If such m, n exist, replace bx with $mx + nx$, and factor by grouping.
- If such m, n don't exist, $ax^2 + bx + c$ has no binomial factors, and so is prime, unless theres a GCF you missed.

Example: Factor $6x^2 - 7x - 20$ by grouping:

We must find m, n such that:

- $mn = (6)(-20) = -120$ so m and n have **opposite signs** and
- $m + n = -7$ so **the negative number has larger absolute value** $|-| > |+|$

This is the same process we use with the monic trinomials, except we will now do something else with m and n when we find them. While you *could* factor a monic trinomial by grouping, it is a complete waste of time.

We list all of the pairs of integers with product -120, where the negative number has larger absolute value, and see if any have a sum of -7, starting with m=1, working up through all the numbers that evenly divide 24:

m	n	$m+n$	
1	-120	-119	
2	-60	–58	
3	-40	-37	
4	-30	-26	
5	-24	-19	
6	-20	-14	
8	-15	**-7**	← aha!
10	-12	-2	

Now, we can factor by grouping:

$$6x^2 + -7x - 20 = 6x^2 + 8x - 15x - 20 = 2x(3x + 4) - 5(3x + 4) = (3x + 4)(2x - 5)$$

3.2.2 derivation - for the curious only!

Multiplying the factors on the right side, we see that:

$$ax^2 + bx + c = (px + q)(rx + s) = prx^2 + psx + qrx + qs = prx^2 + (ps + qr)x + qs$$

So $a = pr$, $b = ps + qr$, and $c = qs$. Notice that $ac = prqs$ contains all four unknown integers, and so does $b = ps + qr$. We can rearrange $ac = psqr = (ps)(qr)$.

Now, if we represent ps with the new variable m and qr with n, we can reformulate our problem of finding p, q, r, s as the following:

Find integers m, n such that $mn = ac$ and $m + n = b$. If they exist, we may factor thusly:

$$ax^2 + bx + c = ax^2 + mx + nx + c = prx^2 + psx + qrx + qs = px(rx + s) + q(rx + s) = (rx + s)(px + q)$$