A few challenging problems

1) Find all functions \( f \) such that \( f' \) is continuous and

\[
[f(x)]^2 = 100 + \int_{0}^{x} \{ |f(t)|^2 + [f'(t)]^2 \} dt
\]

for all real \( x \).

2) Let \( f \) be a function with the property that \( f(0) = 1 \), \( f'(0) = 1 \), and \( f(a + b) = f(a)f(b) \) for all real numbers \( a \) and \( b \). Find \( f \).

3) Find all functions that satisfy the equation

\[
\left( \int f(x)dx \right) \left( \int \frac{dx}{f(x)} \right) = -1
\]

4) Find the curve that passes through the point \((3, 2)\) and has the property that if the tangent line is drawn at any point \( P \) on the curve, then the part of the tangent line that lies in the first quadrant is bisected at \( P \).

5) Let \( f \) be a positive real-valued differentiable function. Let \( f'(x) > f(x) \) for all \( x \). For what integers \( k \) must there exist an integer \( N \) such that \( f(x) > e^{kx} \) for all \( x > N \).

6) Let \( f \) be a twice-differentiable function that satisfies

\[
f(x) + f''(x) = -xg(x)f'(x)
\]

where \( g(x) > 0 \) for all \( x \). Prove that \( |f(x)| \) is bounded.

7) Let \( f \) be a real-valued function with a continuous third derivative such that \( f(x), f'(x), f''(x), f'''(x) \) are positive for all \( x \). Suppose that \( f'''(x) \leq f(x) \) for all \( x \). Show that \( f'(x) < 2f(x) \) for all \( x \).