1) The probability density of X is given by

$$f(x) = \begin{cases} \dfrac{xe^{\frac{-x^2}{2\sigma^2}}}{\sigma^2} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

where $\sigma > 0$. Calculate $E(X)$.

$$E(X) = \frac{1}{\sigma^2}\int_0^\infty x^2 e^{\frac{-x^2}{2\sigma^2}}\,dx.$$

There are many ways to compute this integral. The easiest way is to consider the variance of a Normal variable with mean 0 and variance σ^2. The variance is given by

$$\sigma^2 = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^\infty x^2 e^{\frac{-x^2}{2\sigma^2}}\,dx,$$

and so,

$$\sigma^2 = \frac{2}{\sigma\sqrt{2\pi}}\int_0^\infty x^2 e^{\frac{-x^2}{2\sigma^2}}\,dx.$$

It now follows that

$$\sigma^3\sqrt{\frac{\pi}{2}} = \int_0^\infty x^2 e^{\frac{-x^2}{2\sigma^2}}\,dx.$$

So the mean of X is $\sigma\sqrt{\frac{\pi}{2}}$.

Another approach would be to use the equation

$$\Gamma(\alpha) = 2^{1-\alpha}\int_0^\infty z^{2\alpha-1}e^{-z^2/2}\,dz.$$

First make the substitution $z = \frac{x}{\sigma}$ and let $\alpha = \frac{3}{2}$. So

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{\sqrt{2}}\int_0^\infty \frac{x^2}{\sigma^2}\,e^{\frac{-x^2}{2\sigma^2}}\,\frac{dx}{\sigma}.$$

Now use the fact that $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$ to again obtain

$$\sigma^3\sqrt{\frac{\pi}{2}} = \int_0^\infty x^2 e^{\frac{-x^2}{2\sigma^2}}\,dx.$$

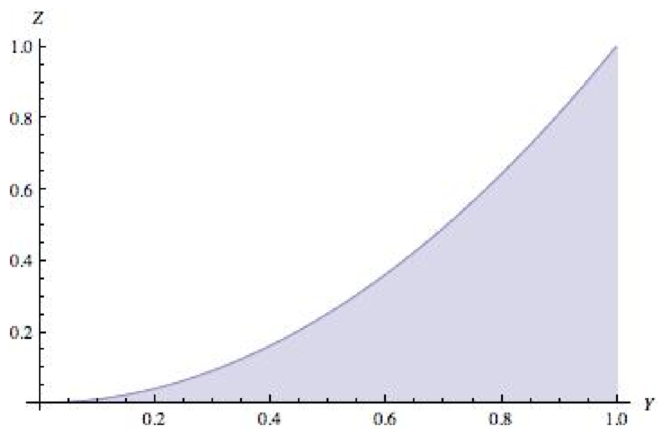
This implies that $E(X) = \sigma\sqrt{\frac{\pi}{2}}$.

2) Let X and Y be independent random variables such that both X and Y have the uniform density on the interval $(0, 1)$. Find the probability density of $Z = XY^2$ (10 points).

The joint density function of X and Y is the uniform density over the unit square. We will treat y as the constant variable, apply the transformation technique, and then integrate over y to obtain the density of Z.

This procedure yields the following joint density function of Z and Y.

$$g(z, y) = \begin{cases} \dfrac{1}{y^2} & \text{for } 0 < z < y^2 \text{ and } 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$



The density function of Z is now obtained by integrating over y.

$$\begin{aligned} h(z) &= \int_{\sqrt{z}}^{1} \frac{1}{y^2}\, dy \\ &= -\frac{1}{y}\Big]_{\sqrt{z}}^{1} \\ &= \frac{1}{\sqrt{z}} - 1. \end{aligned}$$

So the density function of Z is

$$h(z) = \begin{cases} \frac{1}{\sqrt{z}} - 1 & \text{for } 0 < z < 1 \\ 0 & \text{elsewhere} \end{cases}$$

3) The automatic opening device of a military cargo parachute has been designed to open when the parachute is 175 m above the ground. Suppose opening altitude actually has a normal distribution with mean value 175 m and standard deviation 25 m. Equipment damage will occur if the parachute opens at an altitude of less than 100 m. What is the probability that there is equipment damage to the payload of at least one of five independently dropped parachutes? (10 points)

First, compute the probability of failure. Let Z be the standard normal random variable. We need $P(Z < \frac{100-175}{25}) = P(Z < -3)$, which is approximately 0.0013. So the probability of no damage is 0.9987.

Now the probability that there is equipment damage to the payload of at least one of five independently dropped parachutes is approximately

$$1 - (0.9987)^5.$$

4) The number of accidents on Route X in a week is a Poisson random variable with an average of 3 accidents per week and the number of accidents on Route Y in a week is Poisson with average 5 per week. Assuming that these are independent random variables, find the probability that the total number of accidents on these roads is at least one in the same week (10 points).

The sum of X and Y is Poisson with mean 8. We need to calculate $P(X + Y \geq 1) = 1 - P(X + Y = 0) = 1 - e^{-8}$.

5) A family decides to have children until it has four children of the same gender. Assuming that boys and girls are equally likely, what is the probability distribution function of $X =$ the number of children in the family? (10 points)

The probability distribution function for all possible values is given below.

x	4	5	6	7
$P(x)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{5}{16}$

This is just an application of the Negative Binomial distribution.

6) Answer TRUE or FALSE to the following questions (5 points).
a) The probability distribution function of a Binomial random variable with n trials and probability θ of success converges to the density function of a Normal variable with mean $n\theta$ and standard deviation $n\theta(1-\theta)$ as n goes to infinity. FALSE
b) The difference of two independent Poisson random variables is another Poisson random variable. FALSE
c) The sum of two independent Binomial random variables is another Binomial random variable. FALSE
d) If X and Y are two random variables, $\text{var}(aX - bY) = a^2\text{var}(X) + b^2\text{var}(Y)$. FALSE
e) If $M_X(t)$ is a moment generating function of a random variable X, then $M_X(0) = 1$. TRUE