1.8) Prove that the intersection of any collection of subspaces of \( V \) is a subspace of \( V \).

PROOF. Let \( \Omega \) be an indexing set such that \( U_\alpha \) is a subspace of \( V \), for every \( \alpha \in \Omega \), and let \( I \) be the intersection of these subspaces, that is, \( I = \cap_{\alpha \in \Omega} U_\alpha \).

Since the \( U_\alpha \)'s are all subspaces, \( 0 \in U_\alpha \) for every \( \alpha \in \Omega \), and so, \( 0 \in I \). Let \( x \) and \( y \) belong to \( I \); this means that \( x \) and \( y \) are in every \( U_\alpha \). It follows that \( x + y \in U_\alpha \), for every \( \alpha \in \Omega \), because the subspaces are closed under addition. This demonstrates that \( I \) is closed under addition as \( x + y \in U_\alpha \), for every \( \alpha \in \Omega \), implies that \( x + y \in I \). Now let \( a \in F \) and let \( x \in I \). The \( U_\alpha \)'s, being subspaces, are closed under scalar multiplication, and so \( ax \in U_\alpha \), for every \( \alpha \in \Omega \); therefore, \( ax \in I \) and \( I \) is closed under scalar multiplication.

\( \square \)

1.14) Suppose that \( U \) is a subspace of \( P(F) \) consisting of all polynomials \( p \) of the form \( p(z) = az^2 + bz^5 \), where \( a, b \in F \). Find a subspace \( W \) of \( P(F) \) such that \( P(F) = U \oplus W \).

Observe that, by definition, \( U = \text{span}(z^2, z^5) \), and so, \( U \) is a subspace of \( P(F) \). Also, note that \( (1, z, z^2, z^3, z^4, \ldots) \) is a basis for \( P(F) \). Now let \( W = \text{span}(1, z, z^3, z^4, z^6, z^7, z^8, \ldots) \); this means \( W \) is also a subspace of \( P(F) \).

If \( U \cap W \) contains some nonzero polynomial such as \( az^2 + bz^5 \), where \( a, b \in F \), then we get that \( az^2 + bz^5 \in \text{span}(1, z, z^3, z^4, z^6, z^7, z^8, \ldots) \). This is impossible as \( (1, z, z^2, z^3, z^4, \ldots) \) are linearly independent and so no vector in the list may be expressed as a linear combination of other vectors in the list. It follows that \( U \cap W = \{0\} \).

Now \( U + W \) contains all the vectors in the list \( (1, z, z^2, z^3, z^4, \ldots) \), which is a basis for \( P(F) \), and thus, \( U + W \) contains all linear combinations of the basis vectors. This implies that \( U + W \) contains all vectors in \( P(F) \); so \( P(F) \subset U + W \). Since \( U \) and \( W \) are subspaces of \( P(F) \), \( U + W \) is a subspace of \( P(F) \); so \( U + W \subset P(F) \). The two inclusions of the sets obtained prove that \( U + W = P(F) \). This combined with \( U \cap W = \{0\} \) shows that \( P(F) = U \oplus W \).
2.1) Prove that if \((v_1, v_2, ..., v_n)\) spans \(V\), then so does the list \((v_1 - v_2, v_2 - v_3, ..., v_{n-1} - v_n, v_n)\).

PROOF. Let \(w \in V\). So \(w = a_1v_1 + a_2v_2 + ... + a_nv_n\), for some \(a_1, a_2, ..., a_n \in F\). We can rewrite the expression as \(w = a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + ... + (a_1 + a_2 + ... + a_n)v_n\). This shows that \(w \in \text{span}(v_1 - v_2, v_2 - v_3, ..., v_{n-1} - v_n, v_n)\).

Conversely, suppose that \(v \in \text{span}(v_1 - v_2, v_2 - v_3, ..., v_{n-1} - v_n, v_n)\), that is, \(v = a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + ... + a_nv_n\), for some \(a_1, a_2, ..., a_n \in F\). Then \(v = a_1v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + ... + (a_{n-1} - a_n)v_n\), which is certainly in \(\text{span}(v_1, v_2, ..., v_n)\). This shows that the list \((v_1 - v_2, v_2 - v_3, ..., v_{n-1} - v_n, v_n)\) spans \(V\).

\(\square\)

2.10) Suppose \(V\) is finite dimensional, with \(\dim V = n\). Prove that there exist one-dimensional subspaces \(U_1, U_2, ..., U_n\) of \(V\) such that \(V = U_1 \oplus U_2 \oplus ... \oplus U_n\).

PROOF. Let \((v_1, v_2, ..., v_n)\) be a basis for \(V\), and let \(U_i = \text{span}(v_i)\), for \(i \in \{1, 2, ..., n\}\). Since the \(v_i\)'s are not zero, by definition, the \(U_i\)'s are one-dimensional subspaces of \(V\).

\(U_1 + U_2 + ... + U_n\) contains every vector in the basis, and thus, it contains every vector that can be expressed as a linear combination of the basis elements. So \(V \subset U_1 + U_2 + ... + U_n\), and thus, \(V = U_1 + U_2 + ... + U_n\) since \(U_1 + U_2 + ... + U_n\) is a subspace of \(V\).

Suppose \(v \in U_i \cap U_j\), for \(i, j \in \{1, 2, ..., n\}\) and \(i \neq j\). This implies that \(v = av_i = bv_j\), where \(a, b \in F\). Since \(v_i, v_j\) are linearly independent, we get that \(a = b = 0\), and so \(v = 0\). It follows that \(U_i \cap U_j = \{0\}\), for \(i, j \in \{1, 2, ..., n\}\) and \(i \neq j\). This, together with the fact that \(V = U_1 + U_2 + ... + U_n\), proves that \(V = U_1 \oplus U_2 \oplus ... \oplus U_n\).

\(\square\)

3.16) Suppose that \(U\) and \(V\) are finite-dimensional and \(S \in L(V, W)\), \(T \in L(U, V)\). Prove that \(\dim \text{null}(ST) \leq \dim \text{null}(T) + \dim \text{null}(S)\).

PROOF. First, some notation. Let \(T'\) be the linear map \(T\) restricted to the subspace \(\text{null}(ST)\), that is, \(T'u = Tu\) whenever \(u \in \text{null}(ST)\) and \(T'\) is not defined otherwise.
Observe that if \( u \in \text{null}(T') \), then \( T'u = Tu = 0 \). So \( u \in \text{null}(T) \) and we get the simple inclusion \( \text{null}(T') \subset \text{null}(T) \). This implies that \( \dim \text{null}(T') \leq \dim \text{null}(T) \).

Also, if \( u \in \text{null}(ST) \), then \( (ST)u = 0 = S(Tu) = S(T'u) \). This implies that \( T'u \in \text{null}(S) \), and hence, \( \text{range}(T') \subset \text{null}(S) \). From this we obtain our second inequality \( \dim \text{range}(T') \leq \dim \text{null}(S) \).

Now, the dimension formula, Theorem 3.4, applied to \( \text{null}(ST) \) and \( T' \) says \( \dim \text{null}(ST) = \dim \text{null}(T') + \dim \text{range}(T') \). Combining this with the inequalities \( \dim \text{null}(T') \leq \dim \text{null}(T) \) and \( \dim \text{range}(T') \leq \dim \text{null}(S) \), we get the required result \( \dim \text{null}(ST) \leq \dim \text{null}(T) + \dim \text{null}(S) \).

\( \square \)

5.11) Suppose \( S, T \in L(V) \). Prove that \( ST \) and \( TS \) have the same eigenvalues.

**PROOF.** Let \( \lambda \) be an eigenvalue for \( ST \), that is \( ST(v) = \lambda v \), for some nonzero \( v \in V \). Apply \( T \) to both sides. \( T(ST(v)) = (TS)(Tv) = \lambda Tv \). Now, if \( Tv \neq 0 \), then \( Tv \) is an eigenvector with eigenvalue \( \lambda \) for \( TS \). If \( Tv = 0 \), then \( \lambda = 0 \) and \( T \) is not invertible, and hence, \( TS \) is not invertible, which implies that \( TS \) has zero as an eigenvalue since its null space is not trivial. So in all cases, if \( \lambda \) is an eigenvalue for \( ST \), it is also an eigenvalue for \( TS \). A completely symmetric argument shows that if \( \lambda \) is an eigenvalue for \( TS \), it is also an eigenvalue for \( ST \). It follows that \( ST \) and \( TS \) have the same eigenvalues.

\( \square \)

5.14) Suppose \( V \) is finite-dimensional and \( T \in L(V) \). Prove that \( T \) is a scalar multiple of the identity if and only if \( ST = TS \) for every \( S \in L(V) \).

**PROOF.** If \( T = \lambda I \) for some \( \lambda \in F \), then \( ST = S\lambda I = \lambda SI = \lambda IS = TS \), for any \( S \in L(V) \).

Suppose \( ST = TS \) for every \( S \in L(V) \). Let \((v_1, ..., v_n)\) be a basis. \( Tv_1 = a_1v_1 + ... + a_nv_n \), for some \( a_1, ..., a_n \in F \), because the image of \( v_1 \) is some vector in \( V \), and hence, a linear combination of the basis elements.
Now define a linear map $S$ by describing what it does to the basis elements. $S(v_1) = v_1$ and $S$ sends all other basis vectors to zero. Then look at $ST(v_1) = TS(v_1)$. The left side of the equation is $a_1v_1$ and the right side is $Tv_1$. So $Tv_1 = a_1v_1$. The same argument shows that $Tv_k = a_kv_k$, for $k \in \{1, \ldots, n\}$.

We must show that all these $a_k$’s are the same. Again, define a linear map $S$ by describing what it does to the basis elements. $S(v_1) = v_2, S(v_2) = v_1$ and $S$ sends all other basis vectors to zero. Then look at $ST(v_1 + v_2) = TS(v_1 + v_2)$. The left side of the equation is $a_2v_1 + a_1v_2$ while the right side is $a_1v_1 + a_2v_2$. Since these are equal and because $v_1, v_2$ are linearly independent, $a_1 = a_2$. Similar arguments show that $a_i = a_j$, for $i, j \in \{1, \ldots, n\}$.

Since all these $a_k$’s are the same, we may rename them and call them $\lambda$. So we have shown that $Tv_i = \lambda v_i$, for $i \in \{1, \ldots, n\}$. Let $v$ be a vector in $V$, that is $v = a_1v_1 + \ldots + a_nv_n$, for some $a_1, \ldots, a_n \in F$. Now $Tv = T(a_1v_1 + \ldots + a_nv_n) = \lambda a_1v_1 + \lambda a_2v_2 + \ldots + \lambda a_nv_n = \lambda(a_1v_1 + \ldots + a_nv_n) = \lambda v$. This shows that $T$ is a scalar multiple of the identity.

\[\Box\]

5.21) Suppose $P \in L(V)$ and $P^2 = P$. Prove that $V = \text{null}(P) \oplus \text{range}(P)$.

PROOF. $V$ is a direct sum of $\text{null}(P)$ and $\text{range}(P)$ if the intersection of $\text{null}(P)$ and $\text{range}(P)$ is just zero and every vector in $V$ can be written as a sum of a vector in $\text{null}(P)$ with a vector in $\text{range}(P)$.

Suppose $x$ is in both $\text{null}(P)$ and $\text{range}(P)$. Because $x$ is in $\text{null}(P)$, $P(x) = 0$. But $x$ is also in $\text{range}(P)$ so there is a $y$ in $V$ such that $Py = x$. Since $P^2 = P$, $P^2y = Py$, and also $P^2(y) = P(Py) = Px$. It follows that $x = Py = P^2y = Px = 0$. This gives us that $\text{null}(P) \cap \text{range}(P) = \{0\}$.

Let $v$ be a vector in $V$. Then $v = Pv + (v - Pv)$. Now, $Pv \in \text{range}(P)$, by definition, and $P(v - Pv) = Pv - P^2v = 0$, because $P^2 = P$, implies that $(v - Pv) \in \text{null}(P)$. So every vector in $V$ can be written as a sum of a vector in $\text{null}(P)$ with a vector in $\text{range}(P)$.

Since we have checked these two conditions, we have proved that $V = \text{null}(P) \oplus \text{Range}(P)$. \[\Box\]
6.17) Prove that if $P \in L(V)$, $P^2 = P$, and every vector in $null(P)$ is orthogonal to every vector in $range(P)$, then $P$ is an orthogonal projection.

PROOF. By the previous exercise, $V = null(P) \oplus range(P)$ and since every vector in $null(P)$ is orthogonal to every vector in $range(P)$, $null(P) = (range(P))^\perp$. Now let $v \in V$, and so $v = u + n$, with $u \in range(P)$ and $n \in null(P)$. Also, $v = Pv + (v - Pv)$, with $Pv \in range(P)$ and $(v - Pv) \in null(P)$. Because $V = null(P) \oplus range(P)$, the representation of $v$ as a sum of a vector in $range(P)$ with a vector in $null(P)$ is unique, and therefore, $Pv = u$.

So, by definition, $P$ is a projection of $V$ onto $range(P)$ and an orthogonal projection since $null(P) = (range(P))^\perp$. □

6.2) Suppose $u, v \in V$. Prove that $\langle u, v \rangle = 0$ if and only if $||u|| \leq ||u + av||$ for all $a \in F$.

PROOF. $||u|| \leq ||u + av|| \iff ||u||^2 \leq ||u + av||^2 \iff \langle u, u \rangle \leq \langle u + av, u + av \rangle$

$\iff \langle u, u \rangle \leq \langle u, u \rangle + \overline{a}\langle u, v \rangle + a\overline{\langle u, v \rangle} + a\overline{\langle u, v \rangle} \iff -2Re\{a\langle u, v \rangle\} \leq a\overline{\langle u, v \rangle}\langle v, v \rangle$. We will only work with the last inequality.

If $\langle u, v \rangle = 0$, the inequality holds since the L.H.S is zero and R.H.S is always greater than or equal to zero.

Now suppose that $||u|| \leq ||u + av||$ is true for all $a$ in $F$. So we know that $-2Re\{a\langle u, v \rangle\} \leq a\overline{\langle u, v \rangle}$ also holds for all $a$ in $F$. Suppose $\langle u, v \rangle \neq 0$. Write $a = -b\overline{\langle u, v \rangle}$, where $b$ is a positive real. Note that $-2Re\{a\langle u, v \rangle\} = 2b|\langle u, v \rangle|^2$ and $a\overline{\langle u, v \rangle} = b^2|\langle u, v \rangle|^2\langle v, v \rangle$. So the inequality reduces to $2b|\langle u, v \rangle|^2 \leq b^2|\langle u, v \rangle|^2\langle v, v \rangle$. Since $b$ is positive and $|\langle u, v \rangle|^2 \neq 0$, we can divide by $b|\langle u, v \rangle|^2$, preserving the inequality to get $2 \leq b\langle v, v \rangle$. Now we can make the R.H.S as small as we want by letting $b$ go to zero. When the R.H.S is smaller than 2, we get a contradiction. So $\langle u, v \rangle = 0$.

□

6.18) Prove that if $P \in L(V)$, $P^2 = P$, and $\|Pw\| \leq \|w\|$ for all $w \in V$, then $P$ is an orthogonal projection.
PROOF. By 6.17, it is sufficient to show that every vector in \( \text{null}(P) \) is orthogonal to every vector in \( \text{range}(P) \). First, observe that if \( u \in \text{range}(P) \), then \( Pu = u \). This is because there is some \( x \in V \) such that \( Px = u \), and then \( Pu = P^2x = Px = u \).

Now, let \( v \in \text{null}(P) \) and consider the vector \( w = u + av \), with \( a \in F \). Note that \( Pw = Pu + aPv = u \) and so \( \|Pw\| = \|u\| \). The given inequality \( \|Pw\| \leq \|w\| \) gives us \( \|u\| \leq \|u + av\| \), where \( a \) was some arbitrary scalar. It follows that \( \|u\| \leq \|u + av\| \) for all \( a \in F \). By exercise 6.2, we get that \( u \) and \( v \) are orthogonal. Here \( u \) was some arbitrary vector in \( \text{range}(P) \) and \( v \) was some arbitrary vector in \( \text{null}(P) \). So we may conclude that every vector in \( \text{null}(P) \) is orthogonal to every vector in \( \text{range}(P) \), which is sufficient to show that \( P \) is an orthogonal projection.

\( \square \)