A Curvature Formula

Suppose we have a curve which is not parametrized by arc length and we wish to find curvature in a simple way. In the derivation of a simple formula below, we will asume that the curve is parametrized by t and the arc length parameter is s.

Curvature, $\kappa = |\alpha^{''}(s)|$. Let $T(s)$ be the unit tangent tangent vector, then $\kappa = \left|\frac{dT}{ds}\right|$.

If s and T are functions of t, we may write $\kappa = \left|\frac{dT/dt}{ds/dt}\right|$.

$T = \frac{\alpha^{'}(t)}{|\alpha^{'}(t)|}$. So the numerator becomes $\left|\frac{d}{dt}\left(\frac{\alpha^{'}(t)}{|\alpha^{'}(t)|}\right)\right|$.

Also, $\alpha^{'}(t) = \alpha^{'}(s).\frac{ds}{dt}$, and since $|\alpha^{'}(s)| = 1$, we get $\frac{ds}{dt} = |\alpha^{'}(t)|$.

So our formula becomes

$$\kappa = \frac{\left|\frac{d}{dt}\left(\frac{\alpha^{'}(t)}{|\alpha^{'}(t)|}\right)\right|}{|\alpha^{'}(t)|}$$

We may now use this to solve problems like problem 8, page 23, quite easily.

$\alpha(t) = (t, \cosh(t))$. $\alpha^{'}(t) = (1, \sinh(t))$. $|\alpha^{'}(t)| = \cosh(t)$. So $T = \frac{(1,\sinh(t))}{\cosh(t)}$.

$\left|\frac{d}{dt}\left(\frac{\alpha^{'}(t)}{|\alpha^{'}(t)|}\right)\right| = \left|\left(\frac{\sinh(t)}{\cosh^2(t)}, \frac{1}{\cosh^2(t)}\right)\right| = \frac{1}{\cosh(t)}$.

When we divide by $|\alpha^{'}(t)|$, we get the required curvature $\frac{1}{\cosh^2(t)}$.