A few challenging problems

1) A rectangle with length 2 and width 1 is cut into four smaller rectangles by two lines parallel to the sides. Find the maximum and minimum values of the sum of the squares of the areas of the smaller rectangles.

2) Among all planes that are tangent to the surface $xy^2z^2 = 1$, find the ones that are farthest from the origin.

3) If the ellipse $x^2/a^2 + y^2/b^2 = 1$ is to enclose the circle $x^2 + y^2 = 2y$, what values of a and b minimize the area of the ellipse.

4) Find a simple closed curve C for which the value of the integral

$$\int_C (y^3 - y)dx - 2x^3 dy$$

is maximum.

5) If curl $v = 0$ and div $w = 0$ in a three-dimensional volume V, with $w \cdot n = 0$ on the boundary of a 3-dimensional region S, show that v and w are almost orthogonal, that is, $\int\int\int (v^T w) dV = 0$, integrating over S.

6) Let f be analytic in the entire plane and suppose that $|f(z)| \leq A|z|$, for all z, where A is some positive real number. Show that $f(z) = cz$, where c is some complex number with the property $|c| \leq A$.