A few challenging problems

1) Let $f : [0,1] \to R$ be continuous on $[0,1]$. Evaluate the following limits.

a)

$$\lim_{n\to\infty} \int_0^1 x^n f(x)dx$$

b)

$$\lim_{n\to\infty} n \int_0^1 x^n f(x)dx$$

2) Let $f : R \to R$ be continuous on all of R with the property $|f(x) - f(y)| \geq |x - y|$ for all x and y in R. Show that the image of f is all of R.

3) Let $\{a_n\}$ be a sequence of positive real numbers and suppose $\sum a_n$ converges. Show that the series $\sum \sqrt{a_n a_{n+1}}$ and $\sum \frac{a_n a_{n+1}}{a_n + a_{n+1}}$ also converge.

4) Let C be a collection of *disjoint* crosses in R^2. A cross is a union of two line segments of equal, finite length that meet at their centers, with a segment parallel to the x-axis and the other parallel to the y-axis. Show that C is finite or countably infinite.

5) Evaluate the following limit.

$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + ... + \frac{1}{2n} \right).$$

6) Let $f : R \to R$ and $g : R \to R$ be continuous functions satisfying $f(x+1) = f(x)$ and $g(x+1) = g(x)$ for all x in R. Show that

$$\lim_{n\to\infty} \int_0^1 f(x)g(nx)dx = \int_0^1 f(x)dx \int_0^1 g(x)dx.$$