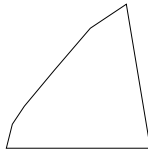


Challenges of the Week
Solutions
Spring Semester 1994-1995

*Challenge of the Week # 1 - January 13 to January 20: An polygon is said to be **convex** if all its interior angles are less than 180° .*

- (A) Determine the maximum number of acute (less than 90°), interior angles that a convex pentagon can have.
(B) Determine the maximum number of acute, interior angles that a convex 1995-gon can have.

This week's problem was solved correctly by Chris Cullop, Joe Nolan, and Kamlesh Parwani. It will be shown that if $n \geq 3$, a convex n -gon contains at most three acute angles. To see that three acute angles are possible in a convex n -gon, consider the following diagram indicating a convex n -gon with three acute angles. (The three acute angles are clearly visible. There are $n - 2$ sides in that portion of the diagram which appears rounded.)



To see that three is the maximum, suppose there is a convex n -gon which has four acute angles at, say, A, B, C and D . These vertices can be joined to form a quadrilateral. Since the original polygon is convex, the quadrilateral is entirely within the original polygon. In particular, each of the interior angles of the quadrilateral at A, B, C , and D is less than the corresponding interior angle of the original polygon. Therefore, each of the interior angles of the quadrilateral is acute. This is impossible since the sum of the interior angles of a quadrilateral is 360° .

Challenge of the Week # 2 - January 20 to January 27: Suppose a, b, c, d are positive real numbers which satisfy

$$\begin{aligned}abcd - a &= 9951 \\abcd - b &= 9591 \\abcd - c &= 5919 \\abcd - d &= 1995\end{aligned}$$

Show that at least one of a, b, c or d is not an integer.

From the papers of Wayne Clark and Kamlesh Parwani. Also solved by Tami Niemerg. By subtracting equations, we have $b - a = 360$, $c - b = 3672$, $d - c = 3924$. Thus all numbers are known, once we know a . First suppose a is even. Then b, c , and d are all even. This is impossible since $abcd - a$ is odd. On the other hand, if a is odd, then each of b, c and d are odd. Since $abcd - a$ is thus the difference of two odd numbers, $abcd - a$ is even. This is also impossible.

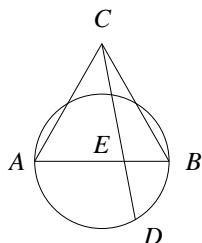
Challenge of the Week # 3 - January 27 to February 3: Let A be a positive integer. Suppose that the sums of the digits of the products $1995 \cdot A$ and $1994 \cdot A$ are equal. Prove that A is divisible by 9.

From the paper of Kamlesh Parwani. Let $1995A = 10^{n-1}x_n + 10^{n-2}x_{n-1} + \dots + 10x_2 + x_1$, where n is the number of digits of the product, x_1 is the units digit of the product, x_2 is the tens digit and so on. Similarly, let $1994A = 10^{m-1}y_m + 10^{m-2}y_{m-1} + \dots + 10y_2 + y_1$. By hypothesis, $x_1 + x_2 + \dots + x_{n-1} = y_1 + y_2 + \dots + y_{m-1}$. Therefore

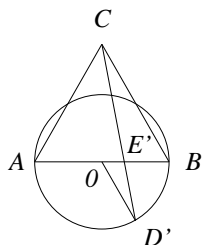
$$\begin{aligned}A &= 1995 \cdot A - 1994 \cdot A \\&= [(10^{n-1} - 1)x_n + (10^{n-2} - 1)x_{n-1} + \dots + (10 - 1)x_2] + [x_1 + x_2 + \dots + x_{n-1}] \\&\quad - [(10^{m-1} - 1)y_m + (10^{m-2} - 1)y_{m-1} + \dots + (10 - 1)y_2] - [y_1 + y_2 + \dots + y_{m-1}] \\&= [(10^{n-1} - 1)x_n + \dots + (10 - 1)x_2] - [(10^{m-1} - 1)y_m + \dots + (10 - 1)y_2].\end{aligned}$$

Since each term of the last sum is divisible by 9, the number A is also divisible by 9.

Challenge of the Week # 4 - February 3 to February 10: Let AB be the diameter of a circle and ABC be an equilateral triangle. Suppose the point E divides the line AB so that $AE = 2 \cdot EB$. If D is the intersection of segment CE , extended, and the circle, as shown, show that the smaller arc AD is half the larger arc ABD .



From the paper of Kamlesh Parwani.



Let O be the center of the circle. Let D' be chosen on the circle, on the opposite side of AB from C , and such that $\angle BOD'$ is a 60° angle. We show that D and D' are the same point.

Let CD' meet AB at E' . Since CD' and AB are straight lines, $\angle OE'D'$ equals $\angle CE'B'$. By construction, $\angle E'BC$ and $\angle E'OD'$ are 60° angles. Therefore $\triangle OD'E'$ is similar to $\triangle BCE'$. In particular, $OD' : BC = OE' : BE'$.

Since $\triangle ABC$ is equilateral and OD' is the radius of the circle with diameter AB , $2 \cdot OD' = AB = BC$. Hence $2 \cdot OE' = BE'$. Thus,

$$2 \cdot EB' = 2 \cdot OB - 2 \cdot OE' = AB - BE' = AE.$$

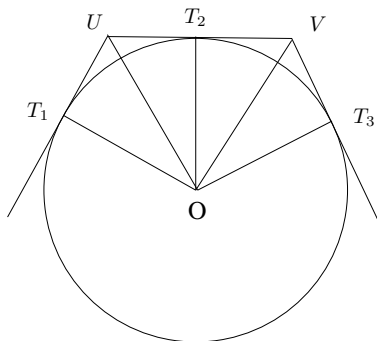
However, by assumption, the point E divides the line AB so that $AE = 2 \cdot EB$. Therefore, $E = E'$ and $D = D'$, as needed.

Challenge of the Week # 5 - February 17 to February 24: A rhombus can be subdivided into two acute triangles by a diagonal joining the two obtuse angles. Given an obtuse triangle, find a way to subdivide it into acute triangles (all angles less than 90°). What is the fewest number of triangles that can be used in such a subdivision? Justify your answers.

No complete solutions were submitted for this problem. There are two steps to the solution. First it must be shown that any obtuse triangle can be divided into 7 acute triangles. Then it must be shown that six triangles do not suffice.

We first prove the following Lemma:

Lemma : Suppose T_1, T_2, T_3 are points on the circle with radius O with UT_1, UT_2 , and VT_3 tangent to the circle with U, T_2, V collinear as shown. If $\angle T_1UT_2$ and $\angle T_2VT_3$ are obtuse, then $\triangle OUV$ is an acute triangle.



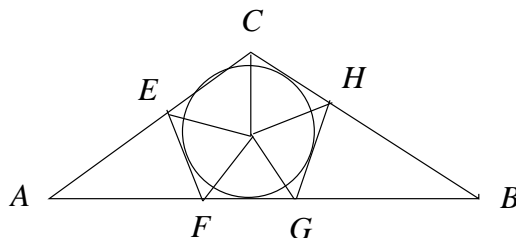
Proof of the Lemma: Since $\triangle OUT_2$ is a right triangle with right angle at T_2 , $\angle OUV < 90^\circ$. Similarly, $\angle OVU < 90^\circ$.

Since OT_1 and OT_2 are radii, they have equal length. Since UT_1 and UT_2 are tangents, they have equal length. Thus $\triangle OUT_1$ is congruent to $\triangle OUT_2$. In particular, $\angle OUT_2 = \frac{1}{2}\angle T_1UT_2$. Since $\angle T_1UT_2$ is obtuse, $\angle OUT_2 > 45^\circ$. Similarly, $\angle OVT_2 > 45^\circ$. Therefore,

$$\angle UOV = 180^\circ - \angle OUT_1 - \angle OUT_2 < 90^\circ.$$

Hence $\triangle OUV$ is an acute triangle.

Proof that 7 triangles suffice. Draw the inscribed circle in the obtuse triangle $\triangle ABC$. Suppose the obtuse angle is at C . Let O be the center of this circle. Construct EF so that $AE = AF$ and EF is tangent to the circle. Also, construct GH so that $GB = BH$ and GH is tangent to the circle, as shown.



Since $\triangle AEF$ is isosceles by construction, all of its angles are acute. Further, $\angle EFG = \angle FEC$ are obtuse. Similarly, $\angle FGH$ and $\angle GHC$ are obtuse. By hypothesis $\angle ACB$ is obtuse. Thus by the lemma, applied 5 times, we see each of the triangles which has O for a vertex is acute. Therefore, an arbitrary obtuse triangle can be divided into 7 acute triangles.

Proof that 7 triangles are necessary. Among all obtuse triangles take one which can be divided into the fewest number of acute triangles and suppose that this obtuse triangle is divided into the minimum number of acute triangles. By way of contradiction, assume that this number is less than 7.

At least one of these triangles must have an edge which meets the obtuse angle, but is not one of the sides of the triangle. If the other vertex of this edge is on one of the sides of the original triangle, then the original triangle is divided into two triangles, at least one of which is a triangle with an obtuse angle. This is impossible since this new obtuse triangle would be subdivided into fewer acute triangles than the original triangle. Thus the other vertex of the edge is interior to the original triangle.

If there are two interior vertices, then at least 5 (acute) triangles meet each of these vertices. Since at most two triangles contain both of these interior vertices, there will have to be at least $2 \cdot 5 - 2 = 8$ triangles in the subdivision, which is a contradiction. Thus there is exactly one interior vertex.

As before, there must be at least five triangles (and five edges) meeting at the interior vertex. Each time an edge is added to these five, another triangle is formed. We will show that there are at least two more edges. One of the five edges goes to the obtuse angle and the other four go either to the other two edges of the triangle or to the edges. Thus at least two of these five edges go to the edges of the triangle. At each of these an obtuse angle is formed and must be subdivided. If two different edges subdivide these angles, then we have at least seven edges and seven

triangles. If the same edge, e subdivides these two angles, then two ends of e lie on different sides of the original triangle and this blocks one of the vertices of the original triangle from being connected to the interior vertices. Thus there is another edge connecting the interior vertices to a side of the original triangle. This forms an obtuse triangle. Hence there must be at least one more edge. This gives at least seven triangles.

Challenge of the Week # 6 - February 24 to March 3: Given a square, find a way to subdivide it into acute triangles (all angles less than 90°). What is the fewest number of triangles that can be used in such a subdivision? Justify your answers.

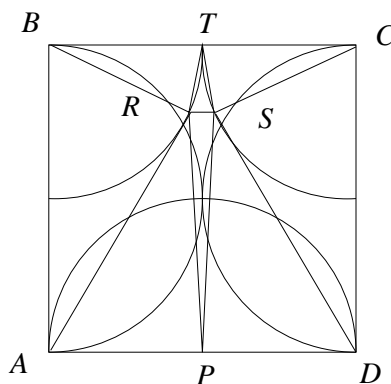
Kamlesh Parwani presented the best solution to this problem. From the Problem of the Week, February 17 to February 24, at least 7 acute triangles are needed in a subdivision of an obtuse triangles. With a little care, this result can be extended to show that at least 7 acute triangles are needed to subdivide a right triangle.

We want to show there are at least 8 acute triangles in a subdivision of a square. Suppose we have a subdivision of a square into the fewest number of acute triangles, which, by way of contradiction, will be assumed to be less than 8. Each of the right angles of the square must be met at least two triangles. Hence at least three edges meet at every corner of the square. Thus every corner meets at least one edge of the subdivision which is not part of the original square. Select one such edge for each corner and label them e_1, e_2, e_3, e_4 . If one of these edges has both of its vertices on the original square, then this edge divides the square into two parts – a right triangle (which requires seven acute triangles in a subdivision) and a quadrilateral or a right triangle. In any case, more than 7 triangles are needed in the subdivision.

If e_1, e_2, e_3, e_4 meet at a common interior vertex, then there is at least one right or obtuse triangle among the four triangles determined by e_1, \dots, e_4 and the sides of the original square. This triangle requires at least 7 acute triangles in a subdivision. Thus the square requires at least 10 acute triangles, a contradiction.

Therefore, each of e_1, \dots, e_4 meets an interior vertex of the subdivision and not all of these vertices are the same. Thus there are at least two interior vertices. Since the square is subdivided into acute triangles, at least 5 of these triangles meet each of these interior vertices. There can be at most two triangles which meet both of these interior vertices. It follows there are at least 8 acute triangles in an subdivision of a square.

Finally, we must show that a square can be divided into 8 acute triangles. Let the square be $ABCD$. Draw semi-circles of diameter AB on three of the sides as shown below. Let PT be the perpendicular bisection with P on AD and T on BC . Let R be a point outside the semicircles with $BR = PT$. Let S be such that RS is parallel to AD and $CS = CT$.



By construction, $\triangle RST$, $\triangle BRT$, $\triangle CST$, and $\triangle PRS$ are all isosceles and hence acute. Also, each of $\angle BAR$, $\angle RBA$, $\angle SDC$, $\angle SCD$, $\angle SPD$, $\angle APR$, $\angle PAR$, and $\angle SDP$ are less than 90° . Since R is outside the semicircle with diameter AB , $\angle ARB$ is acute. Similarly, $\angle DSC$, $\angle DSP$, and $\angle PRA$ are all acute. Hence, the original square can be divided into 8 acute triangles.

Challenge of the Week # 7 - March 1 to March 8: Suppose $x \geq 1$.

(a) Show $\frac{1+x^2}{x} \geq 2$.

(b) Show $\frac{1+x^2+x^4}{x+x^2} \geq \frac{3}{2}$.

From the papers of Xia Lu, Joe Nolan, and Kamlesh Parwani.

(a) Clearly, $(1-x)^2 \geq 0$ for any real value of x . Hence, $1-2x+x^2 \geq 0$. Thus $1+x^2 \geq 2x$. If $x > 0$, this inequality implies

$$\frac{1+x^2}{x} \geq 2.$$

(b) In this case, the given inequality is equivalent to $2x^4 - x^2 - 3x + 2 \geq 0$. Let g be the function defined by $g(x) = 2x^4 - x^2 - 3x + 2$. We must show $g(x) \geq 0$ if $x \geq 1$. Clearly,

$$g'(x) = 8x^3 - 2x - 3$$

$$g''(x) = 24x^2 - 2$$

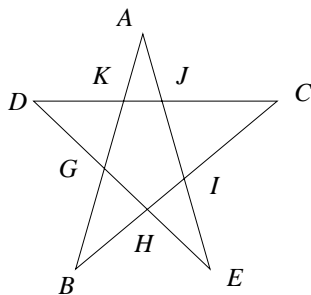
$$g'''(x) = 48x$$

Since $g'''(x) > 0$ for $x \geq 1$, the function g'' is increasing for $x \geq 1$. But $g''(1) = 22$. Hence $g''(x) > 0$ for $x \geq 1$. Thus g' is increasing for $x \geq 1$. Since $g'(1) = 3$, $g'(x) > 0$ for $x \geq 1$. Thus g is an increasing function. Therefore, if $x \geq 1$,

$$g(x) \geq g(1) = 0,$$

as needed.

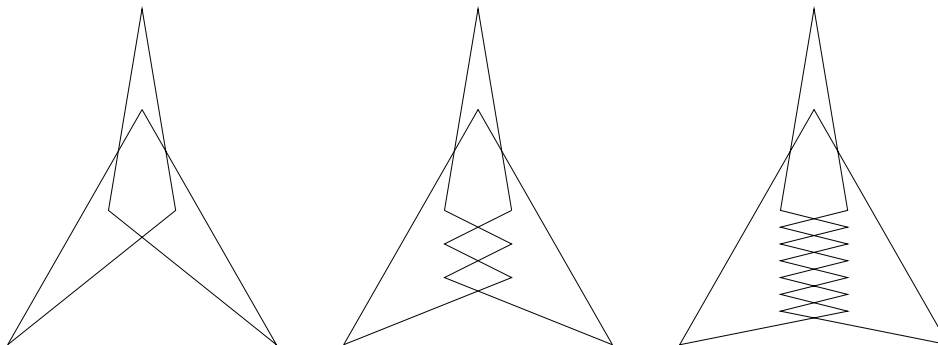
Challenge of the Week # 8 - March 10 to March 24 : A pentagram is formed by joining five points with a single sequence of five, connected line segments so that each line segment meets two other segments in points which are not vertices. Such points will be called double points. In the diagram below, $A, B, C, D,$ and E are the vertices and $G, H, I, J,$ and K are the double-points.



1. Is it possible to find six points which can be joined by a single sequence of six, connected line segments so that each line segment contains exactly one double point?
2. Is it possible to find seven points which can be joined by a single sequence of seven, connected line segments so that each line segment contains exactly one double point?
3. Is it possible to find eight points which can be joined by a single sequence of eight, connected line segments so that each line segment contains exactly one double point?

Justify your answers.

From the paper of Xia Lu. As the following diagram shows, it is possible to find sequences of six, ten, and eighteen points with the required property. In fact, it is possible to find a sequence of points for any even integer larger than 4.



It is not possible to find seven points which can be joined by a single cyclic sequence of seven connected line segments, so that each line contains exactly one double point. It takes two line segments to make one double point, and each line segment can only contain one double point. Hence the number of line segments is twice the number of double points. Therefore, if n points are joined by a single cyclic sequence of n connected line segments, n must be even.

Challenge of the Week # 9 - March 24 to March 31: There are three microprocessors, A , B , and C . Each of them reads a card with a pair (m, n) of integers and prints a card with a pair of integers. Microprocessor A prints $(m - n, n)$, microprocessor B prints $(m + n, n)$, and microprocessor C prints (n, m) . Using these three microprocessors, one can, for example, start with a card with $(3, 8)$ on it and end with a card with $(5, 2)$ on it as follows:

$$(3, 8) \xrightarrow{C} (8, 3) \xrightarrow{A} (5, 3) \xrightarrow{A} (2, 3) \xrightarrow{C} (3, 2) \xrightarrow{B} (5, 2).$$

- (a) Starting with $(17, 76)$ obtain $(18, 65)$.
 (b) Is it possible to start with $(19, 95)$ and end with $(17, 76)$? Explain your answer.

Correct solutions were received from Kamlesh Parwani and Xia Lu. Notice that every integer which is a divisor of both m and n is also a divisor of both $m - n$ and n . Therefore, the greatest common divisor of m and n is the same as the greatest common divisor of $m - n$ and n . Similarly, greatest common divisor of m and n is the same as the greatest common divisor of $m + n$ and m . Therefore, even though the microprocessors change the two numbers on the cards, they do not change the greatest common divisor of the two numbers. Since, the greatest common divisor of 19 and 95 is 19 and the greatest common divisor of 17 and 76 is 1, it is not possible to begin with $(19, 95)$ and end with $(17, 76)$.

Challenge of the Week # 10 - March 31 to April 7: Since these problems come out in the week of April Fool's Day, they are a little different.

1. Bob bought a parrot at a pet shop. When he bought it, the owner of the shop said that the parrot would repeat every word that it heard. After a week of trying to get the parrot to talk with no success, Bob took the parrot back to the pet shop and accused the owner of lying. In fact, the owner did not lie. Explain how this could be.
2. A man just came indoors from a heavy rain. He did not have an umbrella or any other protection from the rain so he was completely soaked. Yet no hair on this head was wet. Explain how this could be.
3. A boomerang can be thrown so that it does not hit any other object and yet still returns to the thrower. Is this possible with a ball?
4. Two brothers who are good friends are driving in two cars with one following the other. The brother in front drives off the road and immediately has a terrible accident. The other brother just drives right by and keeps on going (He does not go for help). Explain how this could be.

Correct Solutions were received from Wayne Clark, Xia Lu, Tina Mininni, Joe Nolan, Kamlesh Parwani, and Christine Uecker.

1. The parrot was deaf.
2. The man was bald.
3. Yes. Throw the ball straight up in the air.
4. The two brothers were driving in a race.

Challenge of the Week # 11 - April 7 to April 14: Starting with 7719, we write a long string of digits

77194159947999 ...

where each digit after the first four is found by taking the last digit of the sum of the preceding four terms. Show that 7720 never occurs as four consecutive digits in this sequence.

Correct solutions were received from Jan Brandt, Xia Lu, Travis Martin, and Tami Niemerg.

Instead of writing down the numbers in the sequence, we only write down the parity of the numbers, i.e. whether the number is even (E) or odd (O). The sequence is then

OOOOE OOOOE OOOOE OOOOE OOOOE ...

This sequence continues forever in the same pattern. The number 7720 never occurs in this sequence since its parity pattern is OOEE.

Note: It can be shown that it will take 1560 steps for the original sequence of numbers to begin repeating!