Solution - Challenge of the Week

Challenge of the Week # 7 - February 29 to March 7, 2007

Does there exist a positive integer n, whose square equals

$$7^{198} + (2^{200} \cdot 7^{99}) + 2^{398}?$$

If yes, is n a prime number of a composite number? Justify your answer

This week's challenge was solved by Matt Niemerg. A submission was also received from Marcus Waller.

By factoring

$$7^{198} + (2^{200} \cdot 7^{99}) + 2^{398} = (7^{99} + 2^{199})^2$$

we see that the number is a perfect square.

It remains to show $7^{99} + 2^{199}$ is not a prime. Now,

$$7^{99} - 1 = (7-1)(7^{98} + 7^{96} + \cdots + 7 + 1)$$

while

$$2^{199} + 1 = (2+1)(2^{198} - 2^{196} + \cdots - 2 + 1).$$

Thus, $7^{99} - 1$ and $2^{199} + 1$ are multiples of 3. Therefore, their sum, $7^{99} + 2^{199}$, is a multiple of 3 and thus it is not a prime.