

Challenges of the Week
Solutions
Spring Semester 2006-2007

Challenge of the Week # 1 - January 12 to January 19: Find all positive integers which, when written in base 10, have all of the following properties:

- (a) *The digits of the integer, when added together, give a sum of 100,*
- (b) *the squares of the digits of the integer, when added together, give a sum of 100, and*
- (c) *no two consecutive digits of the integer are equal.*

Justify your answer.

Solved by Holly Bertram, Evan Bright, Kari Donoho, and Steven Kilty. Other submission from Alicia Kuhn. Since the sum of the squares is equal to the sum of the digits, none of the digits can be two or larger. Hence, all of the digits are either zero or one. Since the sum of the digits is 100, there must be 100 ones. Now, no consecutive digits of the number are equal. Hence the digits are alternately one and zero, starting with 1.

Therefore, there are two different numbers. One of them consists of 200 digits, alternately one and zero, beginning with one and ending with zero. The other consists of 199 digits, alternately one and zero, beginning and ending with one.

Challenge of the Week # 2 - January 19 to January 26: There are two neighboring countries, Astoria and Burea. There are direct airline flights between some of the cities of Astoria and some of the cities of Burea. A city in either country is called a "big city" if it is connected by a direct flight to more than half of the cities in the other country. All other cities in the two countries are called "small cities".

Suppose in Astoria, there is a small city which is connected by direct flights to all of the small cities of Burea. Show that every big city in Astoria is connected by a direct flight to at least one big city in Burea.

Solved by Evan Bright, Kelsey DePew, Matt Niemerg, Symon Shmulker and Corey Watkins. The following solution is that given by Corey Watkins The definition of a "big city" is a city that is connected by a direct flight to **more** than half of the cities in the other country. Therefore, if a city is to be considered "small" it can only be connected to half or less of the cities in the neighboring country. If a small city in Astoria is connected by direct flights to all of the small cities in Burea, all of the small cities in Burea can only account for half or less than half of the total number of cities in Burea. Therefore, big cities in Burea account for half or more than half of the total number of cities in Burea. Because the number of big cities in Burea accounts for at least half of the total cities in Burea, any big city in Astoria is by definition connected to at least one of the big cities of Burea. If it were not, then a big city in Astoria couldn't meet the requirement of being connected to over half of the cities in the neighboring Burea.

Challenge of the Week # 3 - January 26 to February 2: Do there exist two triangles, T_1 and T_2 , such that

- (a) *they are not congruent, but*
- (b) *for every side of T_1 , there is a side of T_2 which has the same length and for every side of T_2 , there is a side of T_1 which has the same length?*

Justify your answer.

Completely solved by Kari Donoho. Partially solved by Holly Bertram, Stephen Puricelli, Amber Schmide, and Zac Trendleman. Other submissions from Kelsey DePew, Ken Haggard, and Steve Udowitz. Yes there do. For example, consider the triangle with sides 1, 1, and 1.5 and the triangle with sides 1, 1.5, and 1.5. Triangles with these side lengths exist because all cases of the triangle inequality are satisfied for these two triangles. In this case, there are four different inequalities

$$\begin{aligned}1 + 1 &\geq 1.5 \\1 + 1.5 &\geq 1 \\1 + 1.5 &\geq 1.5 \\1.5 + 1.5 &\geq 1\end{aligned}$$

all of which are satisfied

Challenge of the Week # 4 - February 2 to February 9: Do there exist two triangles, T_1 and T_2 , such that

- (a) *they are not congruent, but*
 (b) *for every pair of sides of T_1 , there is a pair of sides of T_2 such that the sum of lengths of the two sides from T_1 is the same as the sum of the lengths of the two sides from T_2 ?*

Justify your answer.

The only correct solution to this week's challenge was given by Evan Bright. Other submissions came from Kelsey DePew, Chris Dolak, and Amber Schmidt. One way to do this is to take one triangle whose sides have lengths 6, 6 and 8, while the other triangle has sides of lengths 7, 7, and 5.

Challenge of the Week # 5 - February 14 to February 23: In a small forest there are exactly 300 trees. The trees in the forest are either maple trees, oak trees, or chestnut trees. It is the case that no matter how you select 201 different trees from the forest, at least one of 201 will be a maple tree, at least one will be an oak, and at least one will be a chestnut. Rodney, the Ranger, claims that there are exactly 100 oak, 100 maple, and 100 chestnut trees in the forest. Is he correct? Justify your answer.

Amber Goeckner correctly solved this week's challenge. Kari Donoho submitted a partial solution. Chris Dolak also submitted an entry Suppose there are fewer than 100 oak trees. Then there are at least 201 trees which are eight maple or chestnut. This is not possible. Hence there are at most 100 oak trees. Similarly, there are at most 100 maple trees and at most 100 chestnut trees. Since there are 300 trees altogether, it is not possible that there are more than 100 trees of any of the types.

Challenge of the Week # 6 - February 23 to March 2: A 19-digit positive integer, N , has the property that it is divisible by 11 and whenever two non-equal digits of N are interchanged, the resulting number is not divisible by 11. Find the smallest possible N with these properties. Justify your answer.

Chris Dolak submitted a partial solution. Now a 19-digit positive integer $a_{19}10^{19} + a_{18}10^{18} + a_{17}10^{17} + \dots + a_110 + a_0$, is divisible by 11 if and only if the alternating sum

$$-a_{19} + a_{18} - a_{17} + \dots - a_1 + a_0$$

is divisible by 11. If two of the digits $a_{19}, a_{17}, \dots, a_1$ are unequal, then interchanging them would result in a number which is still divisible by 11. This is not possible. Hence, there is a digit a such that

$$a = a_{19} = a_{17} = \dots = a_1.$$

Similarly, there is a digit b such that

$$b = a_{18} = a_{16} = \dots = a_0.$$

Therefore, the alternating digit sum of this number is $-10a + 9b$. This number is to be a multiple of 11. In this case, $-10a + 9b + 11(a - b) = a - 2b$ will also be a multiple of 11. By taking $a = 2$ and $b = 1$ we get the smallest possible 19 digit number, 2,121,212,121,212,121,212.

Challenge of the Week # 7 - March 23 to March 30: At a certain company, some of the people shake hands with others. No one shakes hands with a person more than once. And no one shakes hands with herself or himself. It is discovered that there is one person who made exactly one handshake, one who made exactly two handshakes, one who made exactly three hand shakes, and so on until one who made exactly 10 handshakes. Determine the smallest possible number of people at this company. Justify your answer.

Evan Bright correctly solved this week's challenge. Other submissions came from Kelsey DePew, Chris Dolak, and Amber Schmidt Since no one shook hands with himself and no person shook hands with another person more than once, there must be at least 11 people. It is possible that there are only 11 people. Let the individuals be A, B, C, D, E, F, G, H, I, J, K.

A shakes hands with B, C, D, \dots K, B shakes hands with C, D, \dots K, C shakes hands with D, \dots , K, \dots , and J shakes hands with K. This means that K shakes hands with A, B, C, \dots , J.

Challenge of the Week # 8 - March 30 to April 6: There are fewer than 40 students in a certain grade school class. The average weight of all the students in the class is 71 pounds. The average weight of the boys is 84 pounds, while the average weight of the girls is only 54 pounds. How many boys and how many girls are in the class? Justify your answer

Correctly solved by Mark Bingham, Tim Comp and Tim Russell. Partial solutions were received from Chris Dolak and Tim Dudek. Let G be the number of girls and B be the number of boys. The total weight of the boys is $84B$. The total weight of the girls is $54G$. Thus, the average weight of all students is

$$\frac{84B + 54G}{B + G} = 71.$$

This means $84B + 54G = 71B + 71G$ or $13B = 17G$. Thus the number of boys is a multiple of 17 and the number of girls is a multiple of 13. That is, $B = 17k_1$ and $G = 13k_2$ where k_1, k_2 are at least one. But $B + G \leq 40$ so $17k_1 + 13k_2 \leq 40$ and

$$17(k_1 - 1) + 13(k_2 - 1) \leq 40 - 30 = 10.$$

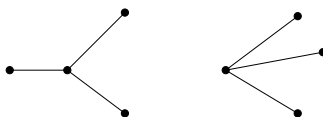
Since $k_1 - 1$ and $k_2 - 1$ are non-negative integers, it must be the case that they are both equal to zero. Hence, there are 17 boys and 13 girls

Challenge of the Week # 9 - April 6 to April 13: Ten quarters are placed into two flexible coin purses. The first purse contains A quarters and the second contains B quarters. Can it happen that the ratio A to B equals 10 to 1? Can it equal 5 to 1? Can it equal 2 to 1? Justify your answers.

Correct solutions were submitted by Jody Alford and Tim Russell. Other submissions from Mark Bingham, Chris Dolak, Ted Effertz, Tyler Funk, Mathew Shafer, and Kate Welden Since the two purses are flexible, one can fit inside the other. Suppose the two purses are called the inner purse and the outer purse. If 1 coin is placed in the inner purse, 9 coins in the outer purse and then the inner purse is placed inside the outer purse, there are 10 coins in the outer purse and 1 in the inner. Similarly, placing 2 in the inner purse, 8 in the outer purse and then placing the inner purse inside the outer gives a ratio of 5 to 1. The last ratio is also possible. Place 5 in the outer purse, the rest in the inner purse and then place the inner purse inside the outer purse.

Challenge of the Week # 10 - April 13 to April 20: In a city, there are six parks. There are streets, which are segments of straight lines, between some of the parks. The following conditions are satisfied:

- Each park is connected by straight line streets with exactly three other parks.*
- None of the straight line streets intersect each other.*
- For every park, the three streets which meet there have the property that two of the streets form an angle, less than 180° , which contains the other street. Thus, the first arrangement of streets, shown below, does not occur. The second arrangement can occur.*



Draw a possible plan for the six parks and their connecting streets.

Correctly solved by Holly Bertram, Mark Bingham, Evan Bright, Chris Dolak, Kari Donoho, Adam Due, Cathleen Fitzgibbon, Stephen Puricelli, Tim Rusell, Corey Watkins, and Kate Welden The following diagram shows two possible plans. These two plans (called graphs) are what is called “isomorphic”. This means the vertices (parks) can be labelled in such a way that two parks are connected in one graph if and only if they are connected in the other. In these two plans, park 1 is connected to parks 2,3,6; park 2 is connected to 1,3,5; park 3 to 1,2,4; park 4 to 3,5,6; park 5 to 2,4,6; and park 6 to 1,4,5.

