Challenges of the Week
Solutions
Spring Semester 2004-2005

Challenge of the Week # 1 - January 21 to January 28: Four persons, A, B, C, and D, are playing poker. At some time during the game; A has as many chips at B, C, and D, together; B has half as many chips as A, C, and D, together; C has 1/N the total number of chips of A, B, and D, together; and D has 1/M the total number of chips of A, B, and C, together; where M and N are different positive integers.

1. *How many poker chips do the four players have together? (There are many possible answers.)*
2. *What is the fewest number of chips that the four players together could have?*

Justify your answers.

Solved by Jill McDonnell, Shannon Price, and Shane Reichart. Other submissions from Nathan Nichols and Monica Will. Let a denote the number of chips that player A has and use b, c, and d for the number of chips the other players have, respectively. At some time, $a = b + c + d$ and $b = \frac{1}{2}(b + c + d)$, $c = \frac{1}{M}(a + b + d)$, and $d = \frac{1}{N}(a + b + c)$. Letting $s = a + b + c + d$, we see that $2a = s$, $3b = s$, $(M + 1)c = s$ and $(N + 1)d = s$. In particular, 2 and 3 divides 6. Therefore, s is a multiple of 6. Also, since M and N are different, C and D do not have the same number of chips. Hence $c + d$ must be at least 3.

Suppose $6 = s$. Then $a = 3$ and $b = 2$. Then means $c + d = 1$. This is not possible.

Suppose $12 = s$. In this case, $a = 6$, $b = 4$ and $c + d = 2$, which is also impossible.

Suppose $18 = s$. In this case, $a = 9$, $b = 6$ and $c + d = 3$. Letting $c = 2$ gives $M = 8$ and $d = 1$ gives $N = 17$.

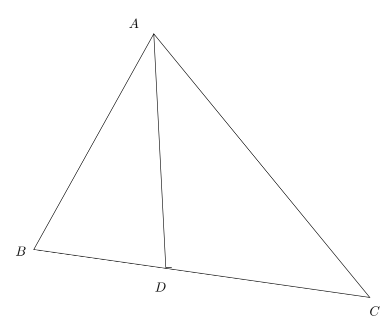
Therefore, the fewest number of chips that the players can have together is seventeen.

Challenge of the Week # 2 - January 28 to February 4: John has 5 slices of pizza and Mary has 3 slices. Robert, who is very hungry, arrives and says that he will pay \$8, if they will divide the pizza equally among the three of them. They agree, so Robert divides the \$8 between John and Mary so they each receive the same amount of money per slice for the pizza they sold to him. How much money did Robert give to Mary? How much money did Robert give to John? Justify your solution.

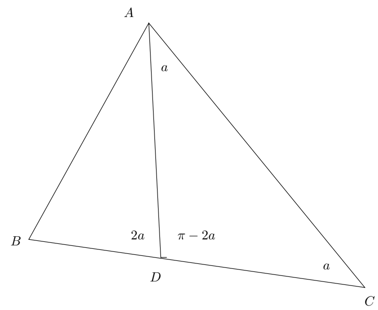
Correct solutions from Kristen Agee, Jill Buchar, Jillian Centers, Colleen Ellis, Karen Klootwyk, Nathan Nichols, Jason Parr, Shannon Price, Shane Reichart, Ted Walk, and Monica Will. Other submissions from Erin Keefe, Anne Sniegowski, and Cailey Swartz. An explanation becomes much clearer it terms of thirds of slices of pizza. John has 15 thirds of slices of pizza. Mary has 9 thirds. Together they have 24 thirds. Each of the three people eats 8 thirds of slices of pizza. John sells 7 of this thirds to Robert. Mary sells one of her thirds. Robert pays one dollar for each third of a slice of pizza that he eats. Therefore, Robert gives 7 dollars to John and one dollar to Mary.

Challenge of the Week # 3 - February 4 to February 11: An triangle isosceles is cut into two smaller isosceles triangles with a single straight cut. What are the possible values for the angles of the triangle? Justify your answer.

Partial solutions were received from Kristen Agee, Shannon Price, Shane Reichart, and Monica Will. Other submissions from Nathan Nichols and Cailey Swartz. If a straight cut divides a triangle into two smaller triangles, the cut must pass through a vertex and the side opposite that vertex. Let $\triangle ABC$ be the triangle, labelled so that the cut goes through A and that $\angle CDA$ is non-acute, as shown:



In particular, $\angle CDA$ is the largest angle of $\triangle ACD$. Since this triangle is isosceles, by assumption, $\angle ACD = \angle DAC$. Letting a be the measure of this angle gives

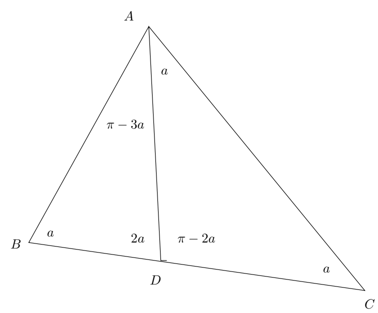


Since $\triangle ACB$ is isosceles and

$$\angle BAC > \angle DAC = \angle ACB,$$

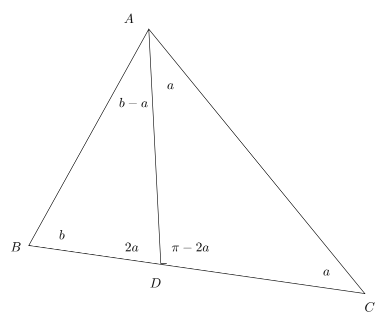
there are two cases: either $\angle CBA = \angle ACB$ or $\angle BAC = \angle CBA$.

Consider first the case that $\angle CBA = \angle ACB$. This gives the following:



Since $\triangle ADB$ is isosceles with $\angle ADB \neq \angle DBA$, either $\angle BAD = \angle DBA$ (and $a = \pi/4$) or $\angle BAD = \angle ADB$ (and $a = \pi/5$).

Next consider the case that $\angle BAC = \angle CBA$. Letting b be the measure of $\angle CBA$. Then $b - a$ is the measure of $\angle BAD$, as shown.

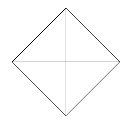


Because $\triangle ADB$ is isosceles with $\angle BAD \neq \angle DBA$, either $\angle ADB = \angle BAD$ or $\angle ADB = \angle DBA$. In the first case,

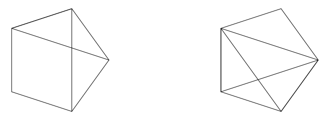
$$2a = \angle ADB = \angle BAD = b - a.$$

Since $2b + a = \pi$, we get that $a = \pi/7$ and $b = 3\pi/7$. A similar argument shows that if $\angle ADB = \angle DBA$, then $a = \pi/5$ and $b = 2\pi/5$.

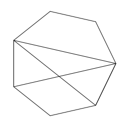
Each of these cases is possible. Consider a square and its diagonals



Two different cases result from considering a regular pentagon and its diagonals.



A final example results from a regular septagon and its diagonals.



Challenge of the Week # 4 - February 11 to February 25: Let N be a positive integer. Consider all of the digits that occur in either the decimal expansion of N or the decimal expansion of $3N$ or in both expansions. Show that one of these digits is either 1, 2 or 9.

Shannon Price submitted a complete solution. Partial solutions were received from Samantha Lininger and Monica Will. Other submission by Anne Sniegowski. Suppose N is a k-digit number so that $10^{k-1} \leq N < 10^k$. If $10^{k-1} \leq N < 3(10)^{k-1}$, the first digit of N is 1 or 2. If $3(10)^{k-1} \leq N < \frac{10}{3}(10)^{k-1}$, the first digit of $3N$ is 9. If $\frac{10}{3}(10)^{k-1} \leq N < 10^k$, $3N$ is a $k+1$-digit number whose first digit is 1 or 2.

Challenge of the Week # 5 - February 25 to March 4: I recently got lost in a very large city. All of the streets in this city go either north-south or east-west. It seemed like I was walking forever. I started down one street and wandered around making left and right turns at random, never going down the same street twice and never crossing my path, until I eventually returned to the starting point, going in the same direction that I started out. I counted. I made a total of 100 left turns in my journey. How many right turns could I have made? Justify your answer.

Solutions from Jamie McGhee and Monica Will. Partial solutions submitted by Shannon Price and Shane Reichart. Other submissions from Samantha Lininger, Nathan Nichols, Jason Parr, Heather Schroeder and Cailey Swartz Suppose there are R right turns and L left turns in my trip. The path along which I travel forms a polygon with $L+R$ vertices. The sum of the interior angles for such a polygon is $S = ((L+R)-2)180^\circ$.

The polygonal path of my trip divides the city into two parts, the inside and the outside. If the inside of the city is to my left, as I come to a corner, a left turn gives an interior angle of 90° and a right turn gives an interior angle of 270°. If the inside is to my right, the right turns give interior angles of 90° and the left turns give interior angles of 270°. Since the inside is either always to my left or always to my right, either $S = L \cdot 270^\circ + R \cdot 90^\circ$ or $S = R \cdot 270^\circ + L \cdot 90^\circ$.

Combining these these results gives that $R = L \pm 4$. Hence, I made either 96 or 104 right hand turns on my trip.

Challenge of the Week # 6 - March 4 to March 11: The positive integer N can be doubled if the last digit in its decimal expansion is moved to the front of its decimal expansion. The last digit of N is 2. What is the smallest possible value of N. Justify your answer.

Solved by Shannon Price, Shane Reichart, Ben Stow, Ted Walk. Other submission from Nathan Nichols. Suppose N is an m-digit number. The k-th digit of N is the same as the $(k-1)$-st digit of $2N$ for $k = 2, 3, \ldots, m$.

The last digit of N is 2. The last digit of $2N$ must be 4. Therefore, the last two digits of N must be 42. Doubling gives that the last two digits of $2N$ are 84. Hence, the last three digits of N are 842. This process can be continued indefinitely.

N	$\cdots$	8	4	1	0	5	2	6	3	1	5	7	8	9	4	7	3	6	8	4	2
$\times$																				2
$2N$	$\cdots$	8	4	2	1	0	5	2	6	3	1	5	7	8	9	4	7	3	6	8	4

If we want the smallest possible value for N, it is necessary to stop the process when a repeating pattern occurs. Hence the smallest possible value for N is 105,263,157,894,736,842.

Challenge of the Week # 7 - March 25 to April 1: Does there exist a right triangle $\triangle ABC$ with $\angle B = 90^\circ$, base $AC = 10$, and area 30? Justify your answer.

Solved by Jill Buchar, Devin Bucke, Collen Ellis, Darcy Hays, Jamie McGhee, Minelia Mirauete, Shannon Price, Shane Reichart, Heather Schroeder, Ben Stover, and Ted Walk. Additional submissions from Kristen Agee, Jillian Centers, Samantha Lininger, Kimberly Smith, and Monica Will. No such triangle is possible. In fact, we show that if $\triangle ABC$ is a triangle with right angle at B and with $AC = 10$, the area of $\triangle ABC$ is at most 25.

The solution relies on the following result from plane geometry: A right triangle can be inscribed in a semi-circle so that the hypotenuse is the diameter of the circle. Thus, if $\triangle ABC$ is a triangle with $\angle B = 90^\circ$, it is possible to inscribe $\triangle ABC$ in a circle with radius AC. Since A, B, C are on a circle, if M is the midpoint of AC, then $MA = MB = MC = 5$. This means that the shortest distance from B to AC is at most 5. Therefore the area of $\triangle ABC$ is at most 25.

Challenge of the Week # 8 - April 1 to April 8: This is the annual April Fool's Day edition of the Challenge of the Week. Justify each of your answers.

1. *Ten is to Three as Three is to Five as Five is to Four as Four is to Four as Thirteen is to ???*
2. *There are three volumes in a certain trilogy. Each volume has 400 pages. The volumes are arranged in order on a shelf. A worm has eaten from the first page of the first volume to the last page of the third volume. How many pages has the worm eaten through?*
3. *You have a pile of 100 bricks. You need to count out 70 bricks. The bricks are heavy so it takes 1 second to count each brick. Can you count out 70 bricks in less than 1 minute? Can you count out 70 bricks in less than 45 seconds?*
4. *Salvador Dali, the Spanish artist who lived from 1904 to 1989, wrote in his book "The Secret Life of Salvador Dali", "Before the contest I bet that I'll win the contest by drawing a picture and never touching a brush or a pencil to the canvas — and I will always receive first prize". What was Salvador Dali's secret method of drawing?*

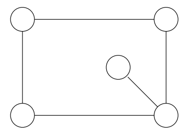
Solutions submitted by Jill Buchar, Nikkole Buchholzer, Devin Bucke, Tyler Buss, Colleen Ellis, Emma Hellstram, Christopher Limbach, Samanta Lininger, Minelia Miravete, Nikki Pisula, Peter Rybchenkov, Heather Schroeder, Kim Smith, Anne Sneigowski, Ben Stover, and Monica Will.

1. The connection between the entries in the list is that if "A is to B", then B is the number of letters in A. There are eight letters in Thirteen so the last entry is "as Thirteen is to Eight".
2. Think about how books are arranged on a shelf. The first page of volume one is on the right of the first volume as you face the shelf. The last page of volume three is on the left of the third volume. The worm has only eaten through the 400 pages of volume two.
3. You can count out 30 bricks in 30 seconds. Seventy bricks remain.
4. Several good answers. "He used a spray gun." "He dipped his brush in paint and then splattered the paint on the canvas without touching the canvas."

Challenge of the Week # 9 - April 9 to April 15: At a certain very fancy store they sell expensive letters to be used in spelling out numbers. The letters are priced individually. Different letters of the alphabet may have different prices, but different copies of the same letter cost the same. It costs \$6 for the letters of "ONE", \$9 for the letters of "TWO", and \$16 for the letters of "ELEVEN". How much does it cost for the letters of "TWELVE"?

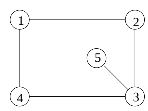
Solved by Kristen Agee, Swarna Latha Bangaru, Holly Bertram, Jill Buchar, Devin Bucke, Katie DePasquale, Samantha Lininger, Minelia Mirauete, Shannon Price, Shane Reichart, Anne Sniegowski, Ben Stover, Greg Taeger, Monica Will, and David Zoerb. Other submissions from Jillian Centers, Jennifer Cooper, Emma Helltram, and Peter Rybchenkov. The letters of "TWELVE" can be obtained by taking the letters of "TWO", adding the letters of "ELEVEN", and then taking away the letters of "ONE". Thus the cost of the letters of "TWELVE" is $9 + 16 - 6 = 19$.

Challenge of the Week # 10 - April 15 to April 22: Five different circles are connected as shown:



1. *How many different color patterns can be obtained if three different colors must be used and no two adjacent circles may be of the same color? Justify your answer.*
2. *How many different color patterns can be obtained if four different colors must be used and no two adjacent circles may be of the same color? Justify your answer.*

Solved by Colleen Ellis. Other submissions from Kristen Agee, James Eiter, Lisa Gembara, Emma Hellstram, Samantha Lininger, Jamie McGhee, and Ben Stover. Let the vertices be numbered as follows:



Consider the problem of determining $C(n)$, the number of different color patterns using n colors subject to the condition that adjacent vertices get different colors, with no requirement that all colors are used. Once the colors on vertices 2 and 4 are determined, it is easy to determine the number of different ways that vertices 1 and 3 can be colored. Then, vertex 5 can be given any color other than the color of vertex 3.

Consider the case that vertices 2 and 4 get the same color. Then, vertices 1 and 3 can be colored in any of the remaining $n-1$ colors. Since there are $n-1$ ways that vertex 5 can be colored, there are a total of $n(n-1)^3$ different color patters in which vertices 2 and 4 can be colored with the same color.

There are $n(n-1)$ different ways that vertices 2 and 4 can be colored in different colors. Then, vertices 1 and 3 can be colored in any of the remaining $n-2$ colors and, as before, there are $n-1$ ways in which vertex 5 can be colored. Combining this with the previous result, we get that

$$C(n) = n(n-1)^3 + n(n-1)^2(n-2)^2 = n(n-1)^2((n-2)^2 + (n-1)).$$

By direct computation, $C(2) = 2$, $C(3) = 36$ and $C(4) = 252$.

If 3 colors must be used, then among the 36 different pattern with three or fewer colors we must eliminate those that involve only two colors. There are three ways to chose two of the three colors and there are 2 different color patterns for each choice of two colors. Hence there are $36 - 6 = 30$ different color patterns if 3 colors must be used.

It is possible to extend the argument of the previous paragraph to the case of 4 colors. In this case, there are $4 \cdot 30$ different patterns which involve exactly three of the four colors and $6 \cdot 2$ different patterns which involve exactly two of the four colors. This means there are $252 - 120 - 12 = 120$ different color patterns.

There is another method for the case of four colors. If four colors must be used, two of the five vertices must be colored the same. These vertices cannot be connected. There are five ways to choose a pair of two vertices which are not connected : 1,3: 1,5; 2,4; 2,5; 4,5. Once a pair has been chosen there are four ways to assign a color to the vertices. For each such assignment there are 6 ways to assign the remaining three colors to the remaining three vertices. Thus, there are 120 different color patterns, as before.