Challenges of the Week
Solutions
Fall Semester 1993-1994

Challenge of the Week # 1 - September 10 to September 17: Compute the following sum and explain how you found the answer.

$$\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \frac{1}{\sqrt{4}+\sqrt{5}} + \cdots + \frac{1}{\sqrt{1992}+\sqrt{1993}}.$$

An exact answer must be given. Decimal approximations are not acceptable.

Solution from the papers of Steve Clark, Matthew T. Holt, Tom King, Chad Mattingly, Joseph R. Nolan, Kamlesh Parwani, Tony Rohr, and Ed Taylor. The trick is to rationalize each fraction involved in the sum. Thus, the original sum equals

$$\frac{1}{1+\sqrt{2}}\,\frac{\sqrt{2}-1}{\sqrt{2}-1} + \frac{1}{\sqrt{2}+\sqrt{3}}\,\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}} + \frac{1}{\sqrt{3}+\sqrt{4}}\,\frac{\sqrt{4}-\sqrt{3}}{\sqrt{4}-\sqrt{3}} + \cdots + \frac{1}{\sqrt{1992}+\sqrt{1993}}\,\frac{\sqrt{1993}-\sqrt{1992}}{\sqrt{1993}-\sqrt{1992}}.$$

This summation now simplifies and telescopes.

$$\frac{\sqrt{2}-1}{2-1} + \frac{\sqrt{3}-\sqrt{2}}{3-2} + \frac{\sqrt{4}-\sqrt{3}}{4-3} + \cdots + \frac{\sqrt{1993}-\sqrt{1992}}{1993-1992}$$
$$= (\sqrt{2}-1) + (\sqrt{3}-\sqrt{2}) + (\sqrt{4}-\sqrt{3}) + \cdots + (\sqrt{1993}-\sqrt{1992}) = \sqrt{1993}-1.$$

Challenge of the Week # 2 - September 17 to September 24: At a party there are six people. Every pair of persons are either friends or enemies. Show that there are either at least three people who are mutual friends or at least three people who are mutual enemies.

Solution from the paper of Joe Nolan. Designate the people as A, B, C, D, E, and F. Since every pair of people are either friends or enemies, person A must have at least three friends or at least three enemies. Without loss of generality, we may assume that A has at least three friends. By relabelling if necessary, assume that B, C, and D are friends of A. If any of the pairs (B, C), (B, D), or (C, D) are pairs of friends, then this pair together with A form a set of three mutual friends. If none of these pairs is a pair of friends, then B, C, and D are mutual enemies.

Challenge of the Week # 3 - September 24 to October 1: Prove or disprove: Given any set of 33 different numbers between 1 and 50, inclusive, there is at least one pair of numbers such that one is twice the other.

Solution from the papers of Tim King, Joe Nolan, Kalesh Parwani, Tony Rohr, Laura Tougaw, and Todd Woods. Let $O = \{1, 3, 5, 7, 9, \ldots, 49\}$ and $E = \{4, 12, 16, 20, 28, 36, 44, 48\}$. Set O contains 25 elements and set B contains 8 elements, so the set $A = B \cup O$ contains 33 elements and no element is twice another.

Challenge of the Week # 4 - October 1 to October 8: Most of the large prime numbers discovered recently have been Mersenne primes. This means they are of the form $2^n - 1$ for n a positive integer. One newspaper, reporting on the largest prime discovered so far, stated that

$$2^{131049} - 1 \quad \textit{is a prime.}$$

Another newspaper reported that

$$2^{132049} - 1 \quad \textit{is a prime.}$$

One of these two reports had a typographical error. Which one was it and why?

Solution from the paper of Chad Mattingly. Also solved by Kamlesh Parwani. Note that $131049 = 3 \cdot 43683$. Therefore,

$$2^{131049} - 1 = (2^{43683})^3 - 1 = (2^{43683} - 1)((2^{43683})^2 + 2^{43683} + 1).$$

Since $2^{131049} - 1$ has positive divisors other than itself and 1, this number is not a prime.

Challenge of the Week # 5 - October 13 to October 22: There are 120 different polynomials of the form

$$Ax^4 + Bx^3 + Cx^2 + Dx + E,$$

where A, B, C, D, E *is an arrangment of* $1, \frac{-1}{2}, \frac{1}{3}, \frac{-2}{3}, \frac{-1}{6}$. *How many of these polynomials have at least one rational zero? Justify your answer.*

Solution from the papers of Kamlesh Parwani and Tony Rohr. Letting $x = 1$ in the given polynomial yields

$$A + B + C + D + E = 1 - \frac{1}{2} + \frac{1}{3} - \frac{2}{3} - \frac{1}{6},$$

since addition is commutative. But $1 - \frac{1}{2} + \frac{1}{3} - \frac{2}{3} - \frac{1}{6} = 0$. Hence the rational number 1 is a zero of all 120 polynomials.

Challenge of the Week # 6 - October 22 to October 29: Show, without using Fermat's Last Theorem, that no positive integer solutions exist for the equation

$$(x + 1)^4 + x^4 = (x + 2)^4.$$

Solution from the papers of Wendy Coplea, Tina Mininni, and Kamlesh Parwani. Also solved by Joe Nolan. Suppose, by way of contradiction, that x is a positive solution to the given equation. Then

$$\begin{aligned} x^4 + (x + 1)^4 - (x + 2)^4 &= x^4 + (x^4 + 4x^3 + 6x^2 + 4x + 1) - (x^4 + 8x^3 + 24x^2 + 32x + 16) \\ &= x^4 - 4x^3 - 18x^2 - 28x - 15 = 0. \end{aligned}$$

That is, $x(x^3 - 4x^2 - 18x - 28) = 15$. Since x and $x^3 - 4x^2 - 18x - 28$ are integers, x must be a factor of 15. Since x is positive, x must be one of $1, 3, 5, 15$. However, it is easy to verify that

$$\begin{aligned} 2^4 + 1^4 &\neq 3^4 \\ 4^4 + 3^4 &\neq 5^4 \\ 6^4 + 5^4 &\neq 7^4 \\ 16^4 + 15^4 &\neq 17^4 \end{aligned}$$

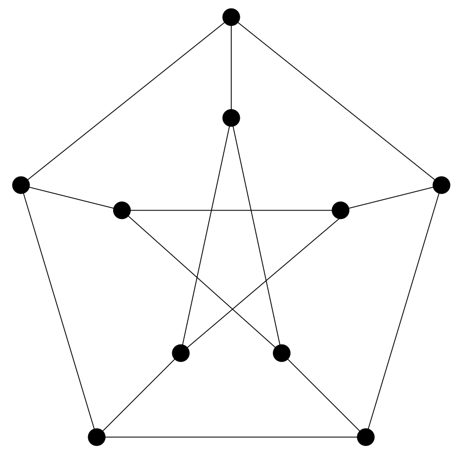
Thus no positive integer solutions exist.

Challenge of the Week # 7 - October 29 to November 5: NonGlobal Airlines serves N *cities in the northern part of a certain southern state. Each of the* N *cities is connected by a non-stop flight to at most three other cities. Further, if* A, B *are any two cities which are not connected by a non-stop flight, there is at least one city,* C, *which is connected by a non-stop flight to* A *and by a non-stop flight to* B. *(Of course,* C *does not have to be the same for every pair* A, B.*)*

Find the maximum possible value for N. *Justify your answer.*

No solutions were received for this problem. From any city A there are non-stop flights to at most three other cities and from those to no more than two additional cities each. Since any city not connected to A by a non-stop flight is connected by a non-stop flight to a city that is connected to A, there are at most $1 + 3 + 3 \cdot 2 = 10$ cities.

A flight system with 10 cities is possible as the following diagram shows.



Challenge of the Week # 8 - November 5 to November 12: A polygon is said to be **convex** if every diagonal of the polygon lies entirely within the polygon. What is the maximum number of acute angles that a a convex polygon of n sides can have? Justify your answer.

Solution from the paper of Kamlesh Parwani. The minimum number of interior angles is 3.

First, we show that it is at most three. Consider a convex polygon of n sides. Pick a point, P, on the interior of the polygon and form n triangles by joining each of the vertices to P. The sum of all the angles of all the triangles is $180°n$. Since the sum of the angles at P is $360°$, the sum of the interior angles of polygon is $180°n - 360°$.

Let x be the number of interior angles which are acute. If S is the sum of the acute, interior angles, then $S < 90°x$. Let T be the sum of the remaining $n - x$ interior angles. Since each is less than $180°$, we have $T < 180°(n - x)$.

Thus

$$(180°n - 360°) - 90°x < (180° - 360°) - S = T < 180°(n - x)$$

This simplifies to $x < 4$, as claimed.

An equilateral triangle is a convex polygon which shows there is a convex polygon with 3 acute, interior angles. (For each value of n, it is possible to modify an equilateral triangle to obtain a convex n-gon with 3 acute, interior angles. Do you see how?)

Challenge of the Week # 9 - November 12 to November 19: How many integers in the sequence

$$1, 11, 111, 1111, 11111, 111111, \cdots$$

are squares of integers? Justify your answer.

Solution from the paper of Kamlesh Parwani. The only square in the sequence is 1. Each element of the sequence is of the form $\frac{10^n - 1}{9}$ for n a natural number. Suppose that $\frac{10^n - 1}{9} = a^2$, where a is a positive integer. Then,

$$\begin{aligned} 10^n - 1 &= 9a^2 \\ 10^n - 2 &= 9a^2 - 1 \\ 2(5 \cdot 10^{n-1} - 1) &= (3a - 1)(3a + 1). \end{aligned}$$

Since $3a - 1$ and $3a + 1$ differ by two they are either both even or both odd. Since their product is even, there is an integer b such that $2b = 3a - 1$. Therefore

$$2(5 \cdot 10^{n-1} - 1) = (2b)(2b + 2)$$

and

$$(5 \cdot 10^{n-1} - 1) = 2b(b + 1).$$

But $5 \cdot 10^{n-1} - 1$ is odd if n is greater than 1. It follows that the only square in the sequence is the first term.