The Fifth Eastern Illinois University
Undergraduate Problem Solving Competition
Solutions
1992 - 1993

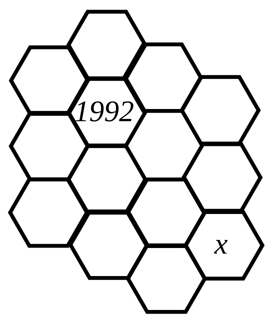
1. *Find all positive integers n and k such that n^n has k digits and k^k has n digits.*

Solution: Since n^n has k digits, then $10^{k-1} \le n^n < 10^k$. Similarly, since k^k has n digits, $10^{n-1} \le k^k < 10^n$. If $k < n$, then

$$n^n < 10^k \le 10^{n-1} \le k^k.$$

This is contradictory to the assumption that $k < n$. Similarly, the assumption $n < k$ will lead to a contradiction. Hence $n = k$. Therefore, $n^n < 10^n$ and, thus, $n < 10$. By trying $n = k = 1, 2, \ldots, 9$, we find the only possibilities to be $n = k = 1$, $n = k = 8$ and $n = k = 9$.

2. *Hexagons cover the xy-plane as in the following figure. Each hexagon contains a positive integer which is the average of the integers in the six hexagons surrounding it. One hexagon contains 1992. What is the value of x in the figure.*



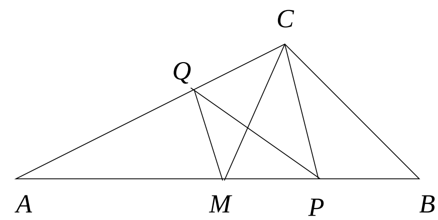
Solution: We claim that all hexagons contain 1992. Suppose some hexagon does not contain 1992. Then, there are two adjacent hexagons which contain y and z, respectively, with $y < z$. The average of the numbers in the six hexagons about the hexagon labelled y must average y and one of these numbers (z) is larger than the average (y). Hence, one of the numbers hexagons adjacent to hexagon containing y must contain a number less than y. Continuing in this way, we get an infinite decreasing sequence of positive integers. This is impossible. Hence, all hexagons must contain 1992.

3. *Two people start with the set of integers $\{1, 2, 3, \ldots, 100, 101\}$. They take turns removing 9 integers from this set. After 11 turns there will be two integers remaining, say a and b with $a > b$. The person with the first turn will receive $a - b$ dollars from the person with the second turn. What is the optimal strategy for each person and what is the payoff with these strategies?*

Solution: Let the person with the first move be A and the other person be B. A always removes the central 9 numbers (so that there are an equal number of numbers larger and smaller than the numbers removed). Thus, at the end of six moves, A will have removed 54 numbers so that b less than all 54 numbers and a more than all 54 numbers. Playing in this way A can get at least a payoff of at least \$55. There is a strategy for B which will result in a payoff of no more than \$55. B always removes the 9 largest numbers. Thus, B can play so that after 5 turns all the the numbers $57, 58, \ldots, 101$ have been removed. Therefore, B can play so that $a < 56$, with a resulting payoff of at most \$55.

4. *Given a point P on side AB of triangle ABC, construct (with straight-edge and compass) a line through P which divides $\triangle ABC$ into two halves.*

Solution: Let M be the midpoint of AB. Clearly, MC divides $\triangle ABC$ into two equal parts. Suppose $M \ne P$. Suppose P lies between M and B, as shown in the following diagram. (The proof if P lies between M and A is similar.)



Construct PC and construct QM parallel to PC with Q on side AC. Construct CM and QP. Now $\triangle QMC$ and $\triangle QMP$ have the same area because they have the same height and the same base. Therefore, in terms of areas we have

$$\frac{1}{2}\triangle ABC = \triangle AMC = \triangle AQM + \triangle QMC = \triangle AQM + \triangle QMP = \triangle APQ.$$

Therefore, PQ divides the original triangle into two equal parts.

5. *a) If you draw 9 line segments without lifting your pencil and end at the starting point, so that no three segments meet at the same point, what is the maximum number of intersections? Prove your answer is possible and is a maximum.*

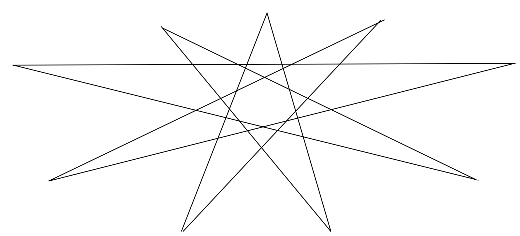
   b) What about 10 line segments?

Solution:

a) Suppose we are drawing n lines segments. Two line segments can either meet at endpoints, intersect, or not meet at all. Consider the intersections in the the order they are created as the figure is drawn. The k-th line segment ($k > 2$) cannot intersect the $k-1$-st line and thus can meet at give at most $k-2$ intersections with previously drawn segments. The n-th line segment meets neither the $(n-1)$-st nor the first line segments. It can meet at most $n-3$ other segments. Thus, the maximum number of intersections is

$$0 + 1 + 2 + \ldots + n - 3 + n - 3 = \frac{(n-2)(n-1)}{2} - 1.$$

For $n = 9$, this upper bound is 27 and can be achieved, as illustrated in the following figure.



b) If n is even, the upper bound, given above, cannot be achieved. In fact, by a long and difficult proof it is possible to show that if n is even the maximum number of intersection is $[n(n-4)/2] + 1$. Thus, for $n = 10$, the maximum is 31. It is interesting (and challenging) to try to draw a figure with this many intersections.

6. *Let a, b, c, d be positive real numbers. Define a sequence $\{s_i\}$, by $s_1 = (a, b, c, d)$ and, if $s_n = (w, x, y, z)$, $n \geq 1$, then $s_{n+1} = (w \cdot x, x \cdot y, y \cdot z, z \cdot w)$. Show that $s_n = s_m$, for some $n \neq m$, if and only if $s_2 = (1, 1, 1, 1)$.*

Solution: Clearly, if $s_2 = (1, 1, 1, 1)$, then $s_3 = (1, 1, 1, 1)$.

Suppose $s_n = s_m$ for $n < m$. Consider the sequence $\{t_i\}$ where t_n is the product of the entries of s_n. Therefore, $t_{n+1} = t_n^2$. It is easy to show by induction that $t_n = (abcd)^{2^{n-1}}$. As $s_n = s_m$, we also have $t_n = t^m$. Therefore,

$$(abcd)^{2^{n-1}} = (abcd)^{2^{m-1}}.$$

Hence $(abcd)^{2^{m-n}} = 1$. Since a, b, c, d are all positive, $abcd = 1$.

Therefore, $s_1 = (a, b, c, d)$, $s_2 = (ab, bc, cd, da)$, $s_3 = (ab^2c, bc^2d, cd^2a, ca^2b)$, and $s_4 = (b^2c^2, c^2d^2, d^2a^2, a^2b^2)$. Another easy induction argument shows that the entries of s_{2n} are a cyclic permutation of $(ab)^{2^{n-1}}$, $(bc)^{2^{n-1}}$, $(cd)^{2^{n-1}}$, and $(da)^{2^{n-1}}$. But the original sequence repeats which implies its terms are bounded. Therefore, bc, cd, da, and ab must each be one or smaller. Since their product is 1, $bc = cd = da = ab = 1$ and the result follows immediately.