The Third Eastern Illinois University
Undergraduate Problem Solving Competition
Solutions
1990 - 1991

1. *There are five houses in a row (east to west), each of a different color and inhabited by people of different nationalities, with different pets and different preferences in beverages and desserts.*

(1) *The Englishman lives in the red house.*

(2) *The Spaniard owns the dog.*

(3) *Coffee is drunk in the green house.*

(4) *The Ukranian drinks tea.*

(5) *The green house in east of the ivory house and next to it.*

(6) *The apple pie lover owns snails.*

(7) *The ice cream lover lives in the yellow house.*

(8) *Milk is drunk in the middle house.*

(9) *The Norwegian lives in the most westerly house.*

(10) *The person who likes cookies lives in the house next to the person with the fox.*

(11) *Ice cream is eaten in the house next to the house where the horse is kept.*

(12) *The cake lover drinks orange juice.*

(13) *The Japanese likes jello.*

(14) *The Norwegian lives next to the blue house.*

Who drinks water and owns the zebra? Explain your answer.

Solution: Let the first house be the most westerly one, the second be the house east of the first, and so forth. The Norwegian lives in the first house (clue 9), so the second house is blue (clue 14). The first house is not red (clue 1) nor is it green or ivory since houses of these colors are next to each other (clue 5). Thus the first house must be yellow. Clues 7, 8 and 11 now give the following partial solution:

	first	second	third	fourth	fifth
Color	yellow	blue			
Nationality	Norwegian				
Animal		horse			
Beverage			milk		
Dessert	ice cream				

Neither the Englishman (clue 1) nor the Spaniard (clue 2) lives in the second house. If the Japanese lives in the second house, then, since the Japanese likes jello (clue 13), the cake lover who likes orange juice (clue 12) lives in the fourth or fifth house. Coffee is also drunk in the fourth or fifth house (clue 3). However, the Ukranian who likes tea must also live in one of the last two houses (clue 4). This is a contradiction and means that the Japanese does not live in the second house. Therefore, by elimination,, the Ukranian lives in the second house. Since clues 3 and 12 show that coffee and orange juice are drunk in the last two houses, the Norwegian must drink water. In summary,

	first	second	third	fourth	fifth
Color	yellow	blue			
Nationality	Norwegian	Ukranian			
Animal		horse			
Beverage	water	tea	milk		
Dessert	ice cream				

The apple pie lover who owns snails is not Norwegian, Ukranian, Japanese (clue 13), nor Spanish (clue 2). Hence she is English and lives in the red house. She does not drink water, tea, coffee (clue 3) nor orange juice (clue 12). Hence she drinks milk and lives in the middle house. This gives us the following:

	first	second	third	fourth	fifth
Color	yellow	blue	red		
Nationality	Norwegian	Ukranian	English		
Animal		horse	snail		
Beverage	water	tea	milk		
Dessert	ice cream		apple pie		

Now jello (Japanese) and cake (with orange juice) are eaten in the last two houses. Thus cookies are eaten in the second house. Since the fox is kept in the next house (clue 10), the fox is kept in the first house. This means that the zebra is kept in one of the last two houses by either the Spanish or the Japanese. Since the Spaniard keeps the dog (clue 2), the zebra is owned by the Japanese (and the Norwegian drinks water).

2. *Determine whether or not there exist 100 distinct lines in the plane having exactly 1990 points of intersection.*

Solution: (from **International Mathematical Olympiads 1975-1985**, edited by Murray S. Klamkin) There do exist 100 distinct lines having exactly 1990 points in common. Consider the 99 lines given by $x = k$, $k = 1, 2, \ldots 73$ and $y = k$, $k = 1, 2, \ldots 26$, and the line $y = 8 - x$. The first 99 lines intersect in $73 \cdot 26$ points. The last line intersects each of the first 99 lines, but 7 of the points, $(1, 7)$, $(2, 6)$, $(3, 5)$, $(4, 4)$, $(5, 3)$, $(6, 2)$, and $(7, 1)$, are also points of intersection of two of the first 99 lines. Therefore, there are $73 \cdot 26 + 99 - 7 = 1990$ points where the lines intersect.

3. *The function f of two variables x and y satisfies*

1. $f(0, y) = y + 1$

2. $f(x + 1, 0) = f(x, 1)$

3. $f(x + 1, y + 1) = f(x, f(x + 1, y))$

for all non-negative integers x and y. Determine $f(4, 1990)$

Solution: The proof consists of a sequence of lemmas:

Lemma 1: $f(1, y) = y + 2, y \geq 0$.

The proof is by induction on y. If $y = 0$, $f(1, 0) = f(0, 1) = 2$. Assume the result for $y = k$, then

$$f(1, k + 1) = f(0, f(1, k)) = f(0, k + 2) = k + 3 = (k + 1) + 2.$$

Hence, by induction, Lemma 1 is true for all non-negative integers y.

Lemma 2: $f(2, y) = 2y + 3, y \geq 0$.

If $y = 0$, $f(2, 0) = f(1, 1) = 3$. Assume, by induction, that the result is true for $y = k$, then

$$f(2, k + 1) = f(2, f(2, k)) = f(1, 2k + 3) = 2k + 5 = 2(k + 1) + 3.$$

The result is thus true by induction.

Lemma 3: $f(3, y) = 10 \cdot 2^y - 3, y \geq 0$.

As before, the result is true for $y = 0$ since $f(3, 0) = f(2, 1) = 7 = 10 \cdot 2^0 - 3$. Assume, by induction, that the result is true for $y = k$, then

$$f(3, k + 1) = f(2, f(3, k)) = f(2, 10 \cdot 2^k - 3) = 2(10 \cdot 2^k - 3) + 3 = 10 \cdot 2^{k+1} - 3$$

and the result follows.

Lemma 4: If $g^{(0)}$ is the function given by

$$g^{(0)}(y) = 10 \cdot 2^y = 3$$

and for $n \geq 1$, $g^{(n)}$ is the function given by

$$g^{(n)}(y) = g^{(0)}\left(g^{(n-1)}(y)\right)$$

then, $f(4, y) = g^{(y)}(1)$.

Again we proceed by induction of y. The result is true for $y = 0$ since, $f(4, 0) = f(3, 1) = 17 = g^{(0)}(1)$. Assume the result for $y = k$, then

$$f(4, k+1) = f(3, f(4, k)) = f(3, g^{(k)}(1)) = 10 \cdot 2^{g^{(k)}(1)} - 3 = g^{(0)}\left(g^{(k)}(1)\right) = g^{(k+1)}(1).$$

Note: This is a very large number!

4. *A* $2 \times 2 \times 12$ *hole in a wall is to be filled with twenty-four* $1 \times 1 \times 2$ *bricks. In how many different ways can this be done if the bricks are indistinguishable?*

Solution: (from **International Mathematical Olympiads 1975-1985**, edited by Murray S. Klamkin) Let T_n be the number of ways of filling up a $2 \times 2 \times n$ hole with $1 \times 1 \times 2$ bricks. We obtain a recurrence relation for T_n.

Let the long axis be vertical. The other two edge directions will be called left-right and forward-backward.

First we count the number of ways of filling up the hole if the bottom layer consists of two bricks lying on their sides. They can lie in two ways, with their long axes in the left-right or the backward-forward direction. The number of these packings is $2T_{n-1}$.

Next consider the packings in which some of the bricks of the bottom layer are oriented upward, but none of the bricks in the bottom two layers extend upward to the next layer. There are five such packings: one in which all four bricks are vertical, one in which the two bricks in front are vertical and the two in back are horizontal, and three more packings of this sort depending upon the placement of the vertical bricks. This give $5T_{n-2}$ more packings.

Now let k be an integer, $2 < k \leq n$. We count the number of ways to fill the hole in which there is at least one brick which extends upward from the j-th to the $j+1$-st layer, $j < k$, but no bricks extend upward from the k-th layer. The bottom layer must be covered by two vertical bricks and one horizontal brick. There is no further choice until we get to the k-th layer: the two empty spaces in each layer must be filled with two vertical bricks. And at the k-th layer, the two empty spaces must be filled with a horizontal brick. Hence there are $4T_{n-k}$ such packings.

This means

$$T_n = 2T_{n-1} + 5T_{n-2} + 4T_{n-3} + \cdots + 4T_0.$$

Also,

$$T_{n-1} = 2T_{n-2} + 5T_{n-3} + 4T_{n-4} + \cdots + 4T_0.$$

Subtracting these two and simplifying, we get $T_n = 3T_{n-1} + 3_{n-2} - T_{n-3}$. Starting with $T_0 = 1$, $T_1 = 2$, and $T_2 = 9$ we get $T_{12} = 4,541,161$.

5. *A box contains p white balls and q black balls and beside the box there is a large pile of black balls. Two balls chosen at random are taken out of the box. If they are of the same color, a black ball from the pile is put into the box; otherwise the white ball is put back into the box. This procedure is repeated until the last two balls are removed from the box and the last ball is put back in. Are there conditions on p and q which ensure the last ball is white? Is so, what are they?*

Solution: The last ball will always be white if p is an odd number. To see this, suppose that p is initially an odd number. After the two balls are selected, there are either $p-2$ white balls, q black balls (if two white balls were selected) or p white balls, $q-1$ black balls (if either two black balls or one of each color were selected). In either case, there remain an odd number of white balls. Thus as the procedure is repeated, the number of white balls remains odd until only one white ball remains.

6. *If A is a subset of the real numbers,* $\mathbb{R}$, *the* **complement** *of A, denoted* $\mathbb{R} - A$, *consists of all elements not in A. The* **closure** *of A, denoted* $\overline{A}$, *is defined by* $x \in \mathbb{R} - A$, *if and only if every open interval containing x intersects A. The offspring of A, consists of the sets* A, $\overline{A}$, $\mathbb{R} - A$, $\overline{\mathbb{R} - A}$, $\mathbb{R} - \overline{A}$, $\mathbb{R} - \overline{\mathbb{R} - A}$, $\overline{\mathbb{R} - \overline{A}}$, ... *For example, if* $A = [0, 1)$, *the offspring are* $[0, 1)$, $[0, 1]$, $\mathbb{R} \setminus [0, 1]$, $\mathbb{R} - [0, 1)$, $\mathbb{R} - (0, 1)$, *and* $(0, 1)$. *Thus* $A = [0, 1)$ *has 6 offspring. Find the maximum number of offspring that a subset of* $\mathbb{R}$ *can have and prove that it is a maximum.*

Solution: (from **International Mathematical Olympiads 1975-1985**, edited by Murray S. Klamkin)

The maximum number of offspring is 14.

Lemma: If B is any subset of $\mathbb{R}$, then $\overline{\mathbb{R} - \overline{\mathbb{R} - \overline{\mathbb{R} - \overline{B}}}} = \overline{\mathbb{R} - \overline{B}}$.

First we show, for C any subset of $\mathbb{R}$, $\overline{\mathbb{R} - \overline{\mathbb{R} - \overline{C}}} \subset \overline{C}$. Now $\mathbb{R} - \overline{C} \subset \overline{\mathbb{R} - \overline{C}}$, so $\mathbb{R} - \overline{\mathbb{R} - \overline{C}} \subset \mathbb{R} - (\mathbb{R} - \overline{C}) = \overline{C}$. The result follows from this inclusion by taking closures.

Letting $C = \mathbb{R} - \overline{B}$, we get $\overline{\mathbb{R} - \overline{\mathbb{R} - \overline{\mathbb{R} - \overline{B}}}} \subset \overline{\mathbb{R} - \overline{B}}$. This completes the proof of the Lemma.

Observe also that if B is any subset, then $\mathbb{R} - (\mathbb{R} - B) = B$ and $\overline{\overline{B}} = B$.

It is now possible to enumerate the possible offspring of a subset A of $\mathbb{R}$. Beginning with $\overline{A}$, the offspring are $\overline{A}$, $\mathbb{R} - \overline{A}$, $\overline{\mathbb{R} - \overline{A}}$, $\mathbb{R} - \overline{\mathbb{R} - \overline{A}}$, $\overline{\mathbb{R} - \overline{\mathbb{R} - \overline{A}}}$, $\mathbb{R} - \overline{\mathbb{R} - \overline{\mathbb{R} - \overline{A}}}$, $\overline{\mathbb{R} - \overline{\mathbb{R} - \overline{\mathbb{R} - \overline{A}}}} = \overline{\mathbb{R} - \overline{A}}$. Beginning with $\mathbb{R} - A$, the offspring are $\mathbb{R} - A$, $\overline{\mathbb{R} - A}$, $\mathbb{R} - \overline{\mathbb{R} - A}$, $\overline{\mathbb{R} - \overline{\mathbb{R} - A}}$, $\mathbb{R} - \overline{\mathbb{R} - \overline{\mathbb{R} - A}}$, $\overline{\mathbb{R} - \overline{\mathbb{R} - \overline{\mathbb{R} - A}}}$, $\mathbb{R} - \overline{\mathbb{R} - \overline{\mathbb{R} - \overline{\mathbb{R} - A}}}$, $\overline{\mathbb{R} - \overline{\mathbb{R} - \overline{\mathbb{R} - \overline{\mathbb{R} - A}}}} = \overline{\mathbb{R} - \overline{\mathbb{R} - A}}$.

A set which has 14 offspring is $A = ((0,1) \cap \mathbb{Q}) \cup (2,3) \cup (3,4) \cup \{5\}$. For this set, $\overline{A} = [0,1] \cup [2,4] \cup \{5\}$, $\mathbb{R} - \overline{A} = (-\infty, 0) \cup (1,2) \cup (4,5) \cup (5,\infty)$, $\overline{\mathbb{R} - \overline{A}} = (-\infty, 0] \cup [1,2] \cup [4,\infty)$, $\mathbb{R} - \overline{\mathbb{R} - \overline{A}} = (0,1) \cup (2,4)$, $\overline{\mathbb{R} - \overline{\mathbb{R} - \overline{A}}} = [0,1] \cup [2,4]$, $\mathbb{R} - \overline{\mathbb{R} - \overline{\mathbb{R} - \overline{A}}} = (-\infty, 0) \cup (1,2) \cup (4,\infty)$, $\mathbb{R} - A = (-\infty, 0] \cup ((0,1) \cap (\mathbb{R} - \mathbb{Q})) \cup [1,2] \cup \{3\} \cup [4,5) \cup (5,\infty)$, $\overline{\mathbb{R} - A} = (-\infty, 2] \cup \{3\} \cup [4,\infty)$, $\mathbb{R} - \overline{\mathbb{R} - A} = (2,3) \cup (3,4)$, $\overline{\mathbb{R} - \overline{\mathbb{R} - A}} = [2,4]$, $\mathbb{R} - \overline{\mathbb{R} - \overline{\mathbb{R} - A}} = (-\infty, 2) \cup (4,\infty)$, $\overline{\mathbb{R} - \overline{\mathbb{R} - \overline{\mathbb{R} - A}}} = (-\infty, 2] \cup [4,\infty)$, $\mathbb{R} - \overline{\mathbb{R} - \overline{\mathbb{R} - \overline{\mathbb{R} - A}}} = (2,4)$.