The First Eastern Illinois University Undergraduate Problem Solving Competition Solutions 1988 - 1989

1. *Find the numerical value of FISH and SHOAL in the following "biological" problem. Different letters stand for different numbers in the set* $\{0, 1, 2, 3, \ldots, 9\}$.

$$FISH + FISH + \ldots + FISH = SHOAL$$

There are 73 FISH in this SHOAL.

Solution: There are two solutions FISH = 0647, SHOAL = 47231 and FISH = 1078, SHOAL=78694. The proof can be split up into cases depending upon the value of S. If S is 9, then SHOAL = 9****, with four digits to be determined. Thus SHOAL $\in [90123, 98765]$, since all the digits have to be different. But $73 \cdots$ FISH = SHOAL, so FISH $\in [1235, 1352]$. Since S = 9, this means FISH = $129*$, i.e. FISH $\in [1290, 1298]$ and SHOAL $\in [94170, 94754]$. In particular, H = 4, FISH = 1294, and SHOAL = 94462. This contradicts the assumption that all digits of SHOAL are different. The proofs in the other cases are similar. The following table summarizes the results with # indicating that a contradiction has been reached.

Value of SHOAL	Range of FISH	Value of FISH	Range of SHOAL	Value of SHOAL	Value of FISH	Value of SHOAL	
0****	[0017,0135]	010*	#				
1****	[0141,0272]	021*	[15549,15987]	15***	0215	15695	#
2****	[0276,0409]	032*	[23652,24017]	#			
3****	[0413,0546]	043*	[31463,32047]	31***	0431	31463	#
		053*	[38763,39347]	38***	0538	39274	#
				39***	0539	39347	#
4****	[0550,0683]	064*	[46793,47377]	47***	0647	47231	
5****	[0687,0820]	075*	[54823,55407]	54***	0754	55042	#
6****	[0824,0957]	086*	[62853,63437]	62***	0862	62926	#
				63***	0863	62999	#
7****	[0960,1094]	097*	[70883,71394]	71***	0971	70883	#
		107*	[78256,78767]	78***	1078	78694	
8****	[1098,1229]	#					
9****	[1235,1352]	129*	[94170,94754]	94***	1294	94462	

2. *Find all positive integers* n *such that* $1! + 2! + 3! + \cdots + n!$ *is the square of an integer. Justify your answer.*

Solution: The only solutions are $n = 1$ and $n = 3$.
These two numbers give solutions since $1! = 1^2$ and $1! + 2! + 3! = 9 = 3^2$.
Note that neither $1! + 2! = 5$ or $1! + 2! + 3! + 4! = 33$ is a square.
Suppose $n \geq 5$. Now, $5! = 120$ and $m!$ is a multiple of 120, for $m \geq 5$. In particular, the units digit of $5! + 6! + \cdots + n!$ is zero. Since the units digit of $1! + 2! + 3! + 4!$ is three, the units digit of $1! + 2! + \cdots + n!$ is also three.

However, the units digit of the square of an integer is the same as the units digit of the square of its units digit. By direct calculation, we see that none of the squares of the digits 0 through 9 is a number which ends in 3. Therefore, if $n \geq 5$, $1! + 2! + \cdots + n!$ is not the square of an integer.

3. *Let* a *and* b *be arbitrary. Describe the set of ordered pairs* (x, y) *of real numbers which simultaneously satisfy*

$$(ax + by)^2 \leq (a^2x + b^2y) \qquad \text{and} \qquad x + y = 1.$$

Solution: Expanding the inequality gives

$$a^2x^2 + 2abxy + b^2y^2 \leq a^2x + b^2y$$

which can be rewritten as

$$2abxy \le a^2x(1-x) + b^2y(1-y).$$

Since $1-x=y$ and $1-y=x$, we have $2abxy \le a^2xy + b^2xy$ or $0 \le (a-b)^2xy$. Therefore $xy \ge 0$. Therefore the ordered pairs satisfying the conditions are

$$\{(x, 1-x) | 0 \le x \le 1\}.$$

4. *A certain set of mathematicians speak in a strange language. Their alphabet has only two letters,* **A** *and* **B**. *They have a rule that in any word* **ABA** *is equivalent to* **B**, *that is,* **ABA** *can be replaced by* **B** *(or vice versa) and the word is considered to be the same. Also they regard* **BAB** *as equivalent to* **AA**. *Thus,* **AAA, ABAB**, *and* **BB** *are to them the same word. Show that there are only a finite number of words in this language. Justify your answer.*

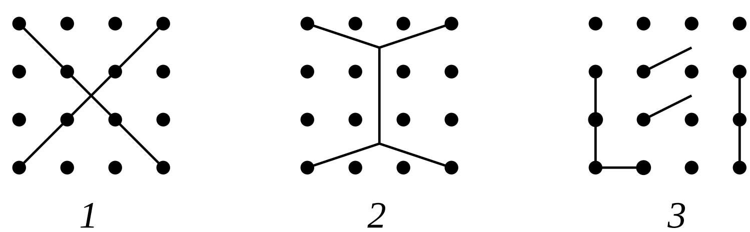
Solution: We enumerate the different words by always replacing a word by a shorter word or by a word of the same length that occurs earlier in lexicographic order. There are two one letter words - A and B. There are four two letter words - AA, AB, BA, and BB. Among the eight four letter words, AAA can be changed to ABAB and then to BB, ABA can be changed to B, BAB can be changed to AA, and BBA can be changed to AAAA and then to ABB. This leaves four three letter words - AAB, ABB, BAA, and BBB. Each of these words can be expanded to a four letter word, but some can be reduced. AABA can be changed to AB, ABBA can be changed to AABB, ABBB can be changed to BBAB and then to BAA, BAAA can be changed to BBB, and BBBA can be changed to BABB and then to AAB. This leaves three that cannot be reduced - AABB, BAAB, and BBBB. There are no five letter words, since whenever a letter is added to one of the reduced four letter words, the resulting word can be reduced. AABBA can be reduced to AAABB and then to BBBB, AABBB can be reduced to ABBAB and then to ABAA, BAABA can be reduced to BAB, BAABB can be reduced to BABBA and then to AABA, BBBBA can be reduced to BBABB and then to BAAB, and finally BBBBB can be changed to AAABAAA and then to AABAA.

5. *If f is a real-valued, continuous function on the set of all real numbers such that $f(0) = 1$ and for which $f(x)f(y) = f(x+y)$ for all real numbers x and y, show that f is differentiable.*

Solution: We claim that $f(x) = f(1)^x$, for all x. It is a straightforward induction argument to show that $f(nx) = f(x)^n$, for all real numbers x and positive integers n. If p and q are positive integers, then $f(1)^p = f(p) = f(q(p/q))) = f(p/q)^q$. Therefore, $f(p/q) = f(1)^{p/q}$. Since $f(0) = 1$ and $f(-x) = 1/f(x)$, for all x, it follows that $f(r) = f(1)^r$ whenever r is a rational number. Suppose x_0 is any real number and $\{p_i/q_i\}$ is a sequence of rational numbers approaching x_0. Since f is continuous,

$$f(x_0) = \lim_{x \to x_0} f(x) = \lim_{i \to \infty} f(p_i/q_i) = \lim_{i \to \infty} f(1)^{p_i/q_i} = f(1)^{x_0}.$$

6. *A collection of line segments contained in a square of side length one inch is said to be an opaque set if every straight line which passes through the square intersects at least one of the line segments. (Think of the line segments as painted walls in a house otherwise made of glass. For an opaque set, a flashlight beam cannot pass through the house.) Examples [1] and [2] are opaque sets while [3] is not.*



An opaque set need not consist of connected line segments (as in [1] and [2]). The length of the opaque set [1] is $2\sqrt{2}$ inches and the length of opaque set [2] is $1+\sqrt{3}$ inches. Find an opaque set whose length is less than $1+\sqrt{3}$ inches. (NOTE: The best solution will be the one with the shortest opaque set.)

Solution: The opaque set below has length $2 + \sqrt{2}/2$ which is less that $1 + \sqrt{3}$. We do not know that the best solution is.