

The First Eastern Illinois University
Undergraduate Problem Solving Competition
1988 - 1989

1. Find the numerical value of *FISH* and *SHOAL* in the following "biological" problem. Different letters stand for different numbers in the set $\{0, 1, 2, 3, \dots, 9\}$.

$$FISH + FISH + \dots + FISH = SHOAL$$

There are 73 *FISH* in this *SHOAL*.

2. Find all positive integers n such that $1! + 2! + 3! + \dots + n!$ is the square of an integer. Justify your answer.

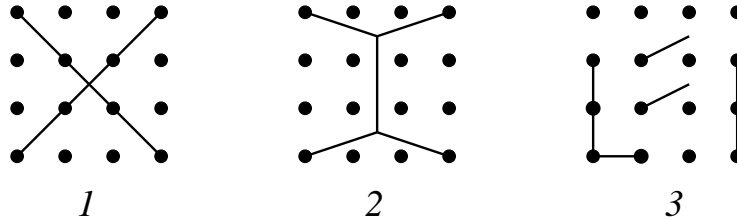
3. Let a and b be arbitrary. Describe the set of ordered pairs (x, y) of real numbers which simultaneously satisfy

$$(ax + by)^2 \leq (a^2x + b^2y) \quad \text{and} \quad x + y = 1.$$

4. A certain set of mathematicians speak in a strange language. Their alphabet has only two letters, **A** and **B**. They have a rule that in any word **ABA** is equivalent to **B**, that is, **ABA** can be replaced by **B** (or vice versa) and the word is considered to be the same. Also they regard **BAB** as equivalent to **AA**. Thus, **AAA**, **ABAB**, and **BB** are to them the same word. Show that there are only a finite number of words in this language. Justify your answer.

5. If f is a real-valued, continuous function on the set of all real numbers such that $f(0) = 1$ and for which $f(x)f(y) = f(x + y)$ for all real numbers x and y , show that f is differentiable.

6. A collection of line segments contained in a square of side length one inch is said to be an opaque set if every straight line which passes through the square intersects at least one of the line segments. (Think of the line segments as painted walls in a house otherwise made of glass. For an opaque set, a flashlight beam cannot pass through the house.) Examples [1] and [2] are opaque sets while [3] is not.



An opaque set need not consist of connected line segments (as in [1] and [2]). The length of the opaque set [1] is $2\sqrt{2}$ inches and the length of opaque set [2] is $1 + \sqrt{3}$ inches. Find an opaque set whose length is less than $1 + \sqrt{3}$ inches. (NOTE: The best solution will be the one with the shortest opaque set.)